

# ON THE STRUCTURAL THEORY OF $\text{II}_1$ FACTORS OF NEGATIVELY CURVED GROUPS, II. ACTIONS BY PRODUCT GROUPS

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**ABSTRACT.** This paper includes a series of structural results for von Neumann algebras arising from measure preserving actions by product groups on probability spaces. Expanding upon the methods used earlier by the first two authors [9], we obtain new examples of strongly solid factors as well as von Neumann algebras with unique or no Cartan subalgebra. For instance we show that every  $\text{II}_1$  factor associated with a weakly amenable group in the class  $\mathcal{S}$  of Ozawa is strongly solid, [35]. There is also the following product version of this result: any maximal abelian  $\star$ -subalgebra of any  $\text{II}_1$  factor associated with a finite product of weakly amenable groups in the class  $\mathcal{S}$  of Ozawa has an amenable normalizing algebra. Finally, pairing some of these results with cocycle superrigidity results from [22], it follows that compact actions by finite products of lattices in  $Sp(n, 1)$ ,  $n \geq 2$ , are virtually  $W^*$ -superrigid.

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## INTRODUCTION

An important motivation in the study of  $\text{II}_1$  factors—in fact, one of von Neumann's original motivations in inventing the subject—is that they provide an analytical and algebraic framework for the representation theory of groups and ergodic

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theory. The usefulness of this observation lies in the fact that classification questions in ergodic theory or representation theory can often be reformulated as questions on the algebraic structure of certain von Neumann algebras, and that such questions may be approached with strategies and techniques beyond those which are available in the standard ergodic or representation-theoretic toolkits. A notable example of the translation of problems from ergodic theory to the theory of von Neumann algebras is the fundamental result of Singer [60] which states that the orbit equivalence class of a free, ergodic, p.m.p. action of a countable discrete group is in one-to-one correspondence with the group of automorphisms of a canonical associated  $\text{II}_1$  factor which preserves a canonical subalgebra. Thus, the problem of characterizing all group actions orbit equivalent to a given one is reduced to the calculation of the symmetry group of some algebraic object.

With such applications in mind, Popa developed in the first half of the last decade a powerful theory for the classification of algebraic structure in  $\text{II}_1$  factors which he termed deformation/rigidity [46, 47, 48]. Popa's techniques rapidly led to the settling of several long-standing problems in the theory of  $\text{II}_1$  factors [46] as well as far-reaching classification results in the orbit equivalence theory of ergodic actions, notably, Popa's cocycle superrigidity theorems [49, 51]. Following Popa's seminal work, the classification of  $\text{II}_1$  factors has witnessed a rebirth. To list some of the major accomplishments which have occurred in the last several years: the cocycle superrigidity theorems of Popa [49, 51] and Ioana [22]; work on the classification of Cartan subalgebras by Ozawa and Popa [41, 42]; the discovery  $W^*$ -superrigid groups and actions with substantial contributions by Ioana, Peterson, Popa, and Vaes, among others, [43, 53, 23, 25, 8, 62]; and the study of various structural properties for von Neumann algebras such as strong solidity initiated by Ozawa and Popa [41, 42] and continued by others [17, 18, 59, 9].

This paper is the continuation of an article [9] by the first two authors. The broad theme of that article was the application of geometric techniques in the context of Popa's deformation/rigidity theory to obtain structural results for  $\text{II}_1$  factors associated to Gromov hyperbolic groups and their actions on measure spaces. This was accomplished in part through the reinterpretation of Ozawa's  $C^*$ -algebraic structural theory of group factors [35, 36] in terms of Peterson's cohomological approach [44] to Popa's deformation/rigidity theory. However, partly for reasons of clarity, there are aspects of Ozawa's theory which were not touched upon in the previous paper—specifically, the use of “small” families of subgroups to unify various structural theorems [5, 57]. The aim of this paper is to incorporate these techniques into the deformation/rigidity approach in [9]. The main applications which will be addressed p.m.p. actions of countable discrete groups  $\Gamma$  which fall into two basic cases: (1)  $\Gamma$  is generated by a pair of subgroups  $(G_1, G_2)$  which are rather “free” with respect to each other (precisely,  $\Gamma$  is relatively hyperbolic to  $\{G_1, G_2\}$ ), or (2)  $\Gamma$  is generated by a pair of “negatively curved” groups  $\{G_1, G_2\}$  with a high degree of commutation. Aside from this we will also be able to sharply generalize most of the results in the previous paper to cover a more general class of groups.

**Statement of results.** The main result of this paper will be the following theorem which improves Theorem B/Theorem 4.1 of [9] in two ways. First, we are able to extend the theorem to the more general class of exact groups which admit proper *arrays* into weakly  $\ell^2$  representations (i.e., bi-exact groups) rather than just proper

quasi-cocycles. Secondly, we are able to deal with groups which are “negatively curved” with respect to a collection of “small” subgroups, which includes, primarily, the widely studied class of relatively hyperbolic groups [3, 13]. The result and its proof are inspired by Ozawa’s general semi-solidity theorem (Theorem 15.1.5 in [5]) and Ozawa and Popa’s Theorem B in [42], viewed through the framework developed by the first two authors in [9].

**Theorem 0.1.** *Let  $\Gamma$  be an exact group, let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a weakly- $\ell^2$  representation and assume that one of the following holds:*

- (1)  $\Gamma$  admits a proper array into  $\mathcal{H}_\pi$  (i.e.,  $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ : see §2); or,
- (2) there exists  $\mathcal{G}$ , a family of subgroups of  $\Gamma$  such that  $\Gamma$  admits a quasi-cocycle which is metrically proper relative to length metric coming from the generating set  $S = \bigcup_{\Sigma \in \mathcal{G}} \Sigma$  (i.e.,  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ : see §2).

Also let  $\Gamma \curvearrowright X$  be a free, ergodic p.m.p. action, denote by  $M = L^\infty(X) \rtimes \Gamma$  the corresponding crossed-product von Neumann algebra, and let  $P \subseteq M$  be any weakly compact embedding with  $P$  diffuse. Then the following holds:

- (1) if  $\Gamma$  satisfies condition (1) above then either the normalizing algebra  $\mathcal{N}_M(P)''$  is amenable or  $P \preceq_M L^\infty(X)$ .
- (2) if  $\Gamma$  satisfies condition (2) above then either the normalizing algebra  $\mathcal{N}_M(P)''$  is amenable or there exists a group  $\Sigma \in \tilde{\mathcal{G}}$  such that  $P \preceq_M L^\infty(X) \rtimes \Sigma$ .

As a consequence any free ergodic weakly compact action [41] of any weakly amenable group  $\Gamma$  in the class  $\mathcal{S}$  of Ozawa [36] gives rise to a von Neumann algebra with unique Cartan subalgebra. Moreover for all these groups as well as all  $\Gamma$  that are hyperbolic relative to a collection of subgroups which are, in some sense, peripheral (cf. §1.2 and §4 in [13]), then  $L\Gamma$  is strongly solid, as the following corollary demonstrates. For instance this will be the case when  $\Gamma$  is any group in the measure equivalence class of an arbitrary limit group in the sense of Sela. These groups should be considered as generalizations of non-uniform lattices in rank one Lie groups, which may admit finitely many cusp subgroups.

**Corollary 0.2.** *Let  $\Gamma$  be a weakly amenable group and let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be an weakly- $\ell^2$  representation such that one of the following holds: either  $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ , or there exists  $\mathcal{G}$ , a family of amenable, malnormal subgroups of  $\Gamma$  such that  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ . If  $\Lambda$  is any ME-subgroup of  $\Gamma$  then  $L\Lambda$  is strongly solid i.e., given any diffuse amenable subalgebra  $A \subseteq L\Lambda$  its normalizing algebra  $\mathcal{N}_{L\Lambda}(A)''$  is still amenable. In particular, every amenable subgroup of  $\Lambda$  has amenable normalizer.*

Our techniques also allow us to obtain structural results for normalizers in direct products of negatively curved groups. Such groups are interesting in that they provide highly tractable examples of groups which exhibit higher-rank (rigid) phenomena (cf. [8, 51, 32, 33]). On the other hand, the next result will show that the structure of their group factors may be reduced to the study of their rank one components (this “rank one” decomposition is algebraically unique by [40]; see also Theorem C in [9]). The result is optimal, though more intricate to state than the previous, since one needs to account for the presence of commutation between the factors.

**Theorem 0.3.** *For every  $i = 1, 2$  let  $\Gamma_i$  be an exact group such that  $\mathcal{RA}(\Gamma_i, \{e\}, \ell^2(\Gamma_i)) \neq \emptyset$ . Let  $(\Gamma_1 \times \Gamma_2) \curvearrowright X$  be a free, ergodic, p.m.p. action and denote by  $M =$*

$L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$  the corresponding crossed-product von Neumann algebra. If  $P \subseteq M$  is any weakly compact embedding with  $P$  diffuse, then one can find projections  $p_0, p_1, p_2, p_3 \in \mathcal{Z}(\mathcal{N}_M(P)' \cap M)$  with  $p_0 + p_1 + p_2 + p_3 = 1$  such that:

- (1)  $\mathcal{N}_M(P)''p_0$  is amenable;
- (2)  $Pp_1 \preceq_M L^\infty(X) \rtimes \Gamma_1$ ;
- (3)  $Pp_2 \preceq_M L^\infty(X) \rtimes \Gamma_2$ ;
- (4)  $Pp_3 \preceq_M L^\infty(X)$ .

The next corollary, for the special case of tensor products of free group factors, is an unpublished result of Ioana and the first author [7]. It would be interesting to know whether the result also holds true for generic higher rank lattices, e.g.  $\mathrm{SL}(3, \mathbb{Z})$ .

**Corollary 0.4.** *Let  $\Gamma_1, \Gamma_2$  be i.c.c. hyperbolic groups and denote by  $M = L\Gamma_1 \bar{\otimes} L\Gamma_2$ . If  $A \subseteq M$  is an amenable subalgebra such that  $A' \cap M$  is amenable (e.g. when  $A$  is either a MASA or an irreducible, amenable subfactor of  $M$ ) then its normalizing algebra  $\mathcal{N}_M(A)''$  is amenable.*

The following corollary is complementary to Corollary 6.2 in [8], which holds for product actions of rigid groups which are sufficiently mixing. Interestingly, for actions between these two extremes, the result is known to fail (Example 2.22 in [33]).

**Corollary 0.5.** *If  $\Gamma_1, \Gamma_2$  are hyperbolic groups with property (T) (e.g.  $\Gamma_i$  lattices in  $\mathrm{Sp}(n, 1)$   $n \geq 2$ ), then any free, ergodic, profinite (or more generally compact) action  $(\Gamma_1 \times \Gamma_2) \curvearrowright X$  is virtually  $W^*$ -superrigid.*

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## 1. POPA'S INTERTWINING TECHNIQUES

We will briefly review the concept of intertwining two subalgebras inside a von Neumann algebra, along with the main technical tools developed by Popa in [46, 47]. Given  $N$  a finite von Neumann algebra, let  $P \subset fNf$ ,  $Q \subset N$  be diffuse subalgebras for some projection  $f \in N$ . We say that *a corner of  $P$  can be intertwined into  $Q$  inside  $N$*  if there exist two non-zero projections  $p \in P$ ,  $q \in Q$ , a non-zero partial isometry  $v \in pNq$ , and a  $*$ -homomorphism  $\psi : pPp \rightarrow qQq$  such that  $v\psi(x) = xv$  for all  $x \in pPp$ . Throughout this paper we denote by  $P \preceq_N Q$  whenever this property holds, and by  $P \not\preceq_N Q$  otherwise. Also the partial isometry  $v$  is called an intertwiner between  $P$  and  $Q$ .

Popa established efficient criteria for the existence of such intertwiners (Theorems 2.1-2.3 in [47]). Particularly useful in concrete applications is the following *analytic* description of absence of intertwiners.

**Theorem 1.1** (Corollary 2.3 in [47]). *Let  $N$  be a von Neumann algebra and let  $P \subset fNf$ ,  $Q \subset N$  be diffuse subalgebras for some projection  $f \in N$ . Then the following are equivalent:*

- (1)  $P \not\preceq_M Q$ .
- (2) *For every finite set  $\mathcal{F} \subset Nf$  and every  $\epsilon > 0$  there exists a unitary  $v \in \mathcal{U}(P)$  such that*

$$\sum_{x, y \in \mathcal{F}} \|E_Q(xvy^*)\|_2^2 \leq \epsilon.$$

As the reader probably noticed by now in the intertwining concept presented before we a priori have no control over the image  $\psi(pPp)$  inside  $qQq$ . When trying to get unitary conjugacy this often becomes a real issue and additional analysis regarding the position of  $\psi(pPp)$  inside  $qQq$  is required. Sometimes the  $\star$ -homomorphism  $\psi$  can be suitably modified to automatically preserve certain properties from the inclusion  $P \subset N$  to the inclusion  $\psi(pPp) \subseteq qQq$ . For instance Ioana showed in Lemma 1.5 of [24] that if  $P \subset N$  is a MASA then  $\psi$  can be chosen so that  $\psi(pPp) \subseteq qQq$  is again a MASA. Applying his argument one can show that  $\psi$  can be chosen to also preserve the irreducibility of inclusion  $P \subset N$ . The precise technical result which will be of essential use to derive some of our main applications is the following:

**Proposition 1.2.** *Let  $N$  be a von Neumann algebra together with subalgebras  $P, Q \subseteq N$  such that  $P' \cap N = \mathbb{C}1$ . If we assume that  $P \preceq_N Q$  then one can find projections  $p \in P$ ,  $q \in Q$ , a  $\star$ -homomorphism  $\phi : pPp \rightarrow qQq$  and a non-zero partial isometry  $v \in qNp$  such that  $\phi(x)v = vx$ , for all  $x \in pPp$ , and  $\phi(pPp)' \cap qQq = \mathbb{C}q$ .*

The proof of this result follows the same recipe as in the Lemma 1.5 of [24] and thus it will be omitted.

We end this section by recalling two important intertwining results which go back to the work of Popa [46, 47]. These results play a very important role in deriving some of our main applications. The first result describes an inclusion of von Neumann algebras where we have complete control over general intertwiners of subalgebras (Theorem 3.1 in [47]). To properly introduce the statement we need a definition. Given an inclusion of countable groups  $\Sigma < \Gamma$  we say that  $\Sigma$  is *malnormal* in  $\Gamma$  if and only if for every  $\gamma \in \Gamma \setminus \Sigma$  we have  $\gamma\Sigma\gamma^{-1} \cap \Sigma$  is finite.

**Proposition 1.3.** *Let  $\Sigma < \Gamma$  be a malnormal group, let  $\Gamma \curvearrowright A$  be a trace preserving action and denote by  $M = A \rtimes \Gamma$  the corresponding crossed product von Neumann algebra. Also let  $p \in A \rtimes \Sigma$  is a projection and suppose that  $P \subseteq p(A \rtimes \Sigma)p$  is a diffuse subalgebra such that  $P \not\preceq_{A \rtimes \Sigma} A$ . If there exist elements  $x, x_1, x_2, \dots, x_n \in M$  such that  $Px \subseteq \sum_i x_i P$  then  $x \in A \rtimes \Sigma$ .*

The second result which will be needed in the sequel is Popa's unitary conjugacy criterion for Cartan subalgebras.

**Theorem 1.4** (Appendix 1 in [46]). *Let  $N$  be a II<sub>1</sub> factor and  $A, B \subset N$  two semiregular MASAs. If  $B_0 \subset B$  is a von Neumann subalgebra such that  $B'_0 \cap N = B$ , and  $B_0 \preceq_N A$ , then there exists a unitary  $u \in N$  such that  $uAu^* = B$ .*

## 2. RELATIVE ARRAYS AND RELATIVE QUASI-COCYCLES

In this section we consider relative versions of the notions of arrays [9] and quasi-cocycles [31, 27, 28] for groups. This will allow us to generalize, from the viewpoint of deformation/rigidity theory, the structural results obtained in [9]. After introducing the definitions, we summarize a few useful properties, relating these with other concepts extant in the literature. In the last part of the section we will present several examples, some of them arising naturally from geometric group theory.

**2.1. Relative arrays.** Assume that  $\Gamma$  is a countable, discrete group together with  $\mathcal{G} = \{\Sigma_i\}_i$ , a family of subgroups of  $\Gamma$  and  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ , a unitary representation. We say that pair  $(\Gamma, \mathcal{G})$  admits a *relative array into  $\mathcal{H}$*  if there exists a map  $r : \Gamma \rightarrow \mathcal{H}$  which satisfies the following conditions:

- (1)  $\pi_\gamma(r(\gamma^{-1})) = -r(\gamma)$  for all  $\gamma \in \Gamma$ ;
- (2) for every  $\gamma \in \Gamma$  we have

$$\sup_{\delta \in \Gamma} \|r(\gamma\delta) - \pi_\gamma(r(\delta))\| = C(\gamma) < \infty;$$

- (3) the map  $\gamma \rightarrow \|r(\gamma)\|$  is proper with respect to  $\mathcal{G}$ , i.e.

$$\lim_{\gamma \rightarrow \infty/\mathcal{G}} \|r(\gamma)\| = \infty.$$

From now on the set of all such relative arrays will be denoted by  $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$ . Notice that when  $\mathcal{G}$  consists of the trivial subgroup only, one recovers the usual notion of arrays as defined in [9].

When considering exact groups, the above notion of relative array into the left regular representation is closely related with the notion of bi-exactness introduced by Ozawa (Definition 15.1.2 in [5]). We are indebted to Narutaka Ozawa for kindly demonstrating to us the direct implication in the following result.

**Proposition 2.1.** *Let  $\Gamma$  be an exact group together with  $\mathcal{G}$  a family of subgroups. Then  $\mathcal{RA}(\Gamma, \mathcal{G}, \ell^2(\Gamma)) \neq \emptyset$  if and only if  $\Gamma$  is bi-exact with respect to  $\mathcal{G}$ .*

*Proof.* The reverse implication can be shown using the same method as in [9] and therefore we only prove the direct implication. So let  $r : \Gamma \rightarrow \ell^2(\Gamma)$  an array relative to the family  $\mathcal{G}$  and denote by  $\pi : \Gamma \rightarrow \mathcal{U}(\ell^2(\Gamma))$  the left regular representation. We still denote by  $\pi : C_\lambda^*(\Gamma) \rightarrow B(\ell^2(\Gamma))$  the induced representation and we denote by  $\phi : C_u^*(\Gamma) \rightarrow B(\ell^2(\Gamma))$  the extension of  $\pi$  to the uniform Roe algebra.

Then we define the map  $\mu : \Gamma \rightarrow S(\ell^\infty(\Gamma))$  by letting

$$\langle \mu(\gamma), f \rangle = \frac{1}{\|r(\gamma)\|^2} \langle \phi(f)r(\gamma), r(\gamma) \rangle,$$

for all  $\gamma \in \Gamma$  and  $f \in \ell^\infty(\Gamma)$ . Here  $S(\ell^\infty(\Gamma))$  denotes the space of states on  $\ell^\infty(\Gamma)$ . Also if we fix  $s, t \in \Gamma$  then denoting by  $C_s = \sup_{\gamma \in \Gamma} \|r(s\gamma) - \pi_s(r(\gamma))\|$  and using the triangle inequality together with the anti-symmetry of the array  $r$  we have that

$$\begin{aligned} \|r(s\gamma t) - \pi_s(r(\gamma))\| &\leq \|r(s\gamma t) - r(s\gamma)\| + \|r(s\gamma) - \pi_s(r(\gamma))\| \\ &= \|-\pi_{s\gamma t}(r(t^{-1}\gamma^{-1}s^{-1})) + \pi_{s\gamma}(r(\gamma^{-1}s^{-1}))\| + C_s \\ (1) \quad &= \|-\pi_t(r(t^{-1}\gamma^{-1}s^{-1})) + r(\gamma^{-1}s^{-1})\| + C_s \\ &\leq C_t + C_s, \end{aligned}$$

for all  $\gamma \in \Gamma$ .

In the remaining part we will use this estimate to show that for all  $s, t \in \Gamma$  we have

$$(2) \quad \lim_{\gamma \rightarrow \infty/\mathcal{G}} \|\mu(s\gamma t) - s.\mu(\gamma)\| = 0,$$

which in turn will give the desired conclusion.

To see this we fix  $s, t, \gamma \in \Gamma$  and  $f \in \ell^\infty(\Gamma)$ . Then applying the triangle inequality in combination with 1 and Cauchy-Schwartz inequality we have

$$\begin{aligned}
& |\langle \mu(s\gamma t), f \rangle - \langle s.\mu(\gamma), f \rangle| \\
\leq & \frac{1}{\|r(s\gamma t)\|^2} |\langle \phi(f)r(s\gamma t), r(s\gamma t) - \pi_s(r(\gamma)) \rangle| + \\
& + \left| \left( \frac{1}{\|r(s\gamma t)\|^2} - \frac{1}{\|r(\gamma)\|^2} \right) \langle \phi(f)r(s\gamma t), \pi_s(r(\gamma)) \rangle \right| + \\
& + \frac{1}{\|r(\gamma)\|^2} |\langle \phi(f)r(s\gamma t) - \pi_s(r(\gamma)), r(\gamma) \rangle| \\
\leq & \frac{C_s + C_t}{\|r(s\gamma t)\|^2} \|\phi(f)r(s\gamma t)\| + \left| \frac{1}{\|r(s\gamma t)\|^2} - \frac{1}{\|r(\gamma)\|^2} \right| \|\phi(f)r(s\gamma t)\| \|r(\gamma)\| + \\
& + \frac{1}{\|r(\gamma)\|} \|\phi(f)r(s\gamma t) - \pi_s(r(\gamma))\| \\
\leq & 2(C_s + C_t) \|f\| \left( \frac{1}{\|r(s\gamma t)\|} + \frac{1}{\|r(\gamma)\|} \right).
\end{aligned}$$

Since  $r$  is assumed to be proper with respect to the set  $\mathcal{G}$  then  $\lim_{\gamma \rightarrow \infty/\mathcal{G}} \|r(s\gamma t)\| = \infty$ ,  $\lim_{\gamma \rightarrow \infty/\mathcal{G}} \|r(\gamma)\| = \infty$  and thus taking the limit in the previous inequality we get 2.  $\square$

A recent result of Popa and Vaes [54] establishes the same result under the weaker assumption that  $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$  for some weakly- $\ell^2$  representation  $\pi$ .

**2.2. Relative quasi-cocycles.** In the same spirit, if  $\Gamma$  is a group together with a family of subgroups  $\mathcal{G} = \{\Sigma_i\}_i$  and a unitary representation  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$ , we say that pair  $(\Gamma, \mathcal{G})$  admits a *relative quasi-cocycle into  $\mathcal{H}$*  if there exists a map  $r : \Gamma \rightarrow \mathcal{H}$  satisfying the following condition:

- (1) there exists a constant  $C > 0$  such that

$$\sup_{\gamma, \delta \in \Gamma} \|r(\gamma\delta) - \pi_\gamma(r(\delta)) - r(\gamma)\| \leq C.$$

- (2) the map  $\gamma \rightarrow \|r(\gamma)\|$  is proper relative to  $\mathcal{G}$ .

From now on, the set of all such relative quasi-cocycles we will denote by  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$ . Using the terminology from [61], it is clear that  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$  is a subset of  $\mathcal{QH}^1(\Gamma, \mathcal{H}_\pi)$  which is stable under scalar multiplication and translation by uniformly bounded maps, without being in general a vector subspace. It is also straight forward that every relative quasi-cocycle is a relative array, i.e., we always have  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \subseteq \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$ . The next proposition summarizes a few basic properties which follow directly from definitions.

**Proposition 2.2.** *For each  $n \in \mathbb{N}$  let  $\mathcal{G}_n$  be a family of subgroups of  $\Gamma$  together with  $\pi_n : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_n)$  a unitary representation. Then we have the following:*

- (1) *If  $\mathcal{G}_1 \subset \mathcal{G}_2$  then  $\mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_{\pi_1}) \subseteq \mathcal{RA}(\Gamma, \mathcal{G}_2, \mathcal{H}_{\pi_1})$ ;*
- (2) *If  $r \in \mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_1)$  and  $c : \Gamma \rightarrow \mathcal{H}_1$  is a uniformly bounded map then  $r + c \in \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H})$ ;*
- (3) *If  $\mathcal{G}_n = \mathcal{G}_1$  and  $\pi_n = \pi_1$  for all  $n$  and there exists a sequence  $r_n \in \mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_{\pi_1})$  with uniformly bounded defects such that  $r_n$  converges to  $r$  uniformly then  $r \in \mathcal{RA}(\Gamma, \mathcal{G}_1, \mathcal{H}_{\pi_1})$ ;*

- (4) Denote by  $\wedge_n \mathcal{G}_n = \{\Sigma_1 \cap \bigcap_{j \neq 1} s_j \Sigma_j s_j^{-1} \mid \Sigma_1 \in \mathcal{G}_1, \Sigma_j \in \mathcal{G}_j, s_j \in \Gamma\}$ . If for every  $n \in \mathbb{N}$  there exists  $c_n > 0$  and  $r_n \in \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_{\pi_n})$  satisfying  $\sum_n c_n^2 \|r_n(\gamma)\|^2 < \infty$  for all  $\gamma \in \Gamma$ , then

$$\mathcal{RA}(\Gamma, \wedge_n \mathcal{G}_n, \oplus \mathcal{H}_{\pi_n}) \neq \emptyset.$$

Cocycles, quasi-cocycles, and arrays combine both geometric and representation-theoretical data in a way that can be used to efficiently extract information about a group's internal structure. For instance, by the same proof as in Proposition 1.5.3 of [9] we can locate centralizers of certain subgroups and, in some cases, even normalizers. This property, generically called the “spectral gap rigidity principle”, is the main intuition for the von Neumann algebraic structural results obtained in the subsequent sections.

**Proposition 2.3.** *Let  $\Gamma$  be a countable group,  $\mathcal{G}$  be a family of subgroups, and  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  a representation such that  $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ . If  $\Lambda < \Gamma$  is a subgroup such that  $1_\Lambda \not\prec \pi$  then its centralizer  $C_\Gamma(\Lambda)$  is small with respect to the family  $\mathcal{G}$ .*

Here are two concrete situations when this happens:  $\Lambda$  has property (T) and the restriction  $\pi|_\Lambda$  has no invariant vectors;  $\Lambda$  is not co-amenable with respect to a subgroup  $\Sigma < \Gamma$  and  $\pi$  is the left semi-regular representation  $\ell^2(\Gamma/\Sigma)$ .

Moreover, if  $\Gamma$  is weakly amenable (for definition see next section),  $\Lambda$  is amenable, and the normalizing group satisfies  $1_{N_\Gamma(\Lambda)} \not\prec \pi$  then  $\Lambda$  is small with respect to  $\mathcal{G}$ .

**Example 2.4.** There are many examples of groups that admit relative quasi-cocycles (arrays) into various representations. First we analyze few examples arising from canonical group constructions:

$\rightsquigarrow$  **Exact sequences.** Let  $L, K, \Gamma$  be groups such that  $0 \rightarrow L \rightarrow K \rightarrow \Gamma \rightarrow 0$  is a short exact sequence. If  $\mathcal{RA}(\Gamma, \{e\}, \ell^2(\Gamma)) \neq \emptyset$  then we have  $\mathcal{RA}(K, \{L\}, \ell^2(K/L)) \neq \emptyset$ .

$\rightsquigarrow$  **Product groups.** Let  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$  be a collection of groups, and denote by  $\Gamma = \Gamma_1 \times \Gamma_2 \times \dots \times \Gamma_n$ . For every  $i = 1, \dots, n$  we denote by  $\hat{\Gamma}_i$  the subgroup of the direct product  $\Gamma$  which consists of all elements whose  $i^{\text{th}}$  coordinate is trivial. Assume that  $\mathcal{G}_i$  is family of subgroups of  $\Gamma_i$  and denote by  $\mathcal{G} = \bigcup_i \{\Lambda \times \hat{\Gamma}_i \mid \Lambda \in \mathcal{G}_i\}$ . If  $\mathcal{RA}(\Gamma_i, \mathcal{G}_i, \mathcal{H}_i) \neq \emptyset$  for all  $i = 1, \dots, n$ , then  $\mathcal{RA}(\Gamma, \mathcal{G}, \otimes_i \mathcal{H}_i) \neq \emptyset$ . For the proof of this fact see Proposition 1.10 in [9]. In particular  $\Gamma_1 \times \Gamma_2$  admits an array into  $\ell^2(\Gamma_1 \times \Gamma_2)$  which is proper with respect to  $\{\Gamma_1, \Gamma_2\}$  whenever  $\Gamma_1$  and  $\Gamma_2$  admit proper array into their left regular representations.

$\rightsquigarrow$  **Semidirect products.** Let  $\Gamma$  and  $A$  be countable discrete groups together with  $\mathcal{G}$  a family of subgroups of  $\Gamma$  and assume that  $\rho : \Gamma \rightarrow \text{Aut}(A)$  is an action by group automorphisms. Let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a unitary representation and define  $\tilde{\pi} : A \rtimes_\rho \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  by letting  $\tilde{\pi}_{a\gamma}(\xi) = \pi_\gamma(\xi)$  for every  $a \in A, \gamma \in \Gamma$  and  $\xi \in \mathcal{H}_\pi$ . If  $c \in \mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi)$  then the formula  $\tilde{c}(a\gamma) = c(\gamma)$  defines an array which belongs to  $\mathcal{RA}(A \rtimes_\rho \Gamma, \{A \rtimes_\rho \Sigma \mid \Sigma \in \mathcal{G}\}, \mathcal{H}_{\tilde{\pi}})$ .

We now look at semidirect products by finite groups. So let  $\Gamma$  be a countable discrete group together with a family of subgroups  $\mathcal{G}$ ,  $\Lambda$  be a finite group, and  $\rho : \Lambda \rightarrow \text{Aut}(\Gamma)$  be an action by automorphisms. It is an exercise for the reader to check that for any  $r \in \mathcal{RA}(\Gamma, \mathcal{G}, \ell^2(\Gamma))$ , the map  $r'(\gamma\alpha) = \frac{1}{|\Lambda|} \sum_{\delta \in \Lambda} \lambda_\delta(r(\rho_{\delta^{-1}}(\gamma)))$  defines an array belonging to  $\mathcal{RA}(\Gamma \rtimes_\rho \Lambda, \mathcal{G}, \ell^2(\Gamma \rtimes_\rho \Lambda))$ , where  $\gamma \in \Gamma, \alpha \in \Lambda$  and



$\lambda$  is the left regular representation on  $\ell^2(\Gamma \rtimes_\rho \Lambda)$ . The defect of  $r'$  will not exceed the defect of  $r$ .

We briefly point out that the above construction admits a slight generalization. Here  $\Lambda$  will denote a locally finite countable discrete group,  $\Lambda = \cup_n \Lambda_n$  with  $\Lambda_n \nearrow$  and  $|\Lambda_n| < \infty$  for all  $n$ . As before, we assume that there exists an action by automorphisms  $\rho : \Lambda \rightarrow \text{Aut}(\Gamma)$ . Given any array  $r \in (\Gamma, \{e\}, \ell^2(\Gamma))$  then, by averaging  $r$  over the orbits of  $\Lambda_n$  as in the previous example and using ultralimit techniques, one can build an array  $r' \in \mathcal{RA}(\Gamma \rtimes \Lambda, \{\Lambda\}, \ell^2(\Gamma \rtimes \Lambda))$ .

$\rightsquigarrow$  **Free products.** Let  $\{\Gamma_n\}_{1 \leq i \leq n}$  be a finite collection of groups. Denote by  $\Gamma = \star_i \Gamma_i$  their free product, and let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  a unitary representation. If for every  $1 \leq i \leq n$  we have  $\mathcal{RA}(\Gamma_i, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ , then the proof of Lemma 5.1 and Theorem 5.3 in [61] show that  $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi^{\oplus n}) \neq \emptyset$ . Note that the when considering arrays proper with respect to families of subgroups, this property become a little more subtle the issue being the properness property. Indeed by Lemma 5.1 from two arrays on two groups we can always canonically construct an array on the free product of the the two groups. However, if the two initial arrays are proper with respect to families of subgroups, through this construction it is not clear wether the new resulting array is proper with respect to any canonical groups but rather just to finite length subsets over the families of groups we started with.

If we assume that  $\Sigma \triangleleft \Gamma_i$  is a common normal subgroup,  $\Gamma = \star_\Sigma \Gamma_i$  is the amalgamated free product over  $\Sigma$ , and for every  $1 \leq i \leq n$  we have  $\mathcal{RA}(\Gamma_i/\Sigma, \{e\}, \ell^2(\Gamma_i/\Sigma)) \neq \emptyset$  then  $\mathcal{RA}(\Gamma, \{\Sigma\}, \ell^2(\Gamma/\Sigma)) \neq \emptyset$ .

$\rightsquigarrow$  **HNN-extensions.** Denote by  $\Gamma = (H, L, \theta)$  the HNN-extension associated with a given inclusion groups  $L < H$  and a monomorphism  $\theta : L \rightarrow H$ . We also assume that  $K \triangleleft H$  is a normal subgroup which contains  $L$  and  $\theta(L)$  and from now on we will denote by  $L_1 = L$ ,  $L_{-1} = \theta(L)$ . The group  $\Gamma$  may be presented as  $\{H, t \mid \theta(\ell) = t\ell t^{-1}, \ell \in L\}$ . By Britton's Lemma, every element  $\gamma \in \Gamma$  has a canonical reduced form  $\gamma = \gamma_0 t^{\varepsilon_1} \gamma_1 t^{\varepsilon_2} \dots \gamma_{n-1} t^{\varepsilon_n} \gamma_n$ , where  $\gamma_i \in H$ ,  $\varepsilon_i \in \{-1, 1\}$  and whenever  $\varepsilon_i \neq \varepsilon_{i+1}$  we have that  $\gamma_i \notin L_{\varepsilon_i}$ , for all  $i = 1, \dots, n-1$ .

Assume that  $q : H/K \rightarrow \ell^2(H/K)$  is an array. By the construction in the first example there exists an array  $c : H \rightarrow \ell^2(H/K)$  which vanishes on  $K$  and moreover  $c \in \mathcal{RA}(H, \{K\}, \ell^2(H/K))$  whenever  $q$  is proper. We can define a map  $r : \Gamma \rightarrow \ell^2(\Gamma/K)$  in the following way: for every  $\gamma = \gamma_0 t^{\varepsilon_1} \gamma_1 t^{\varepsilon_2} \dots \gamma_{n-1} t^{\varepsilon_n} \gamma_n$  and  $s = 0, 1$  we let

$$\begin{aligned} r_q^s(\gamma) &= \lambda_{\gamma_0 t^{\varepsilon_1} \gamma_1 t^{\varepsilon_2} \dots \gamma_{n-1} t^{\varepsilon_n}} c(\gamma_n) + \lambda_{\gamma_0 t^{\varepsilon_1} \gamma_1 t^{\varepsilon_2} \dots \gamma_{n-1}} d^s(t^{\varepsilon_n}) + \\ &\quad + \lambda_{\gamma_0 t^{\varepsilon_1} \gamma_1 t^{\varepsilon_2} \dots \gamma_{n-1} t^{\varepsilon_{n-1}}} c(\gamma_{n-1}) + \lambda_{\gamma_0 t^{\varepsilon_1} \gamma_1 t^{\varepsilon_2} \gamma_{n-2}} d^s(t^{\varepsilon_{n-1}}) + \\ &\quad + \dots + \lambda_{\gamma_0} d^s(t^{\varepsilon_1}) + c(\gamma_0), \end{aligned}$$

where  $d^1(t^\varepsilon) = \delta_{t^\varepsilon K}$  and  $d^0(t^\varepsilon) = 0$  for all  $\varepsilon \in \{-1, 1\}$ . Here  $\lambda$  denotes the left semi-regular representation  $\ell^2(H/K)$ . It is straightforward exercise to see that this map is well defined and it satisfies the array relation. Moreover, when  $q = 0$ , the map is actually a 1-cocycle.

Therefore, applying part (4) in Proposition 2.2 we have that  $r_0^1 \oplus r_q^0$  is an array into  $\ell^2(\Gamma/K) \oplus \ell^2(\Gamma/K)$ . If  $q$  is assumed proper it follows that  $r_0^1 \oplus r_q^0$  is proper with respect to various *subsets* of  $\Gamma$ , e.g. sets of words with finite length over  $t$ 's whose letters from  $H$  are “small” over  $K$ . However to have properness with respect

to subgroups we need to impose additional assumptions on  $K$ . For instance, one may assume that  $L$  and  $\theta(L)$  have finite index in  $K$ , in which case we would have  $r_0^1 \oplus r_q^0 \in \mathcal{RA}(\Gamma, \{K\}, \ell^2(\Gamma/K) \oplus \ell^2(\Gamma/K))$ .

$\rightsquigarrow$  **Inductive limits.** Let  $\Gamma_n \nearrow \Gamma$  be an inductive limit of groups and for each  $n \in \mathbb{N}$  let  $\mathcal{G}_n$  be a family of subgroups of  $\Gamma_n$  such that  $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ . Assume that for each  $n$ , there exists  $r_n \in \mathcal{RA}(\Gamma_n, \mathcal{G}_n, \ell^2(\Gamma_n))$  so that:

- (1)  $\sup_{\gamma \in \Gamma_{\min(n,m)}} \|r_n(\gamma) - r_m(\gamma)\| < \infty$ , for every  $n, m \in \mathbb{N}$ ;
- (2)  $\sup_{n \in \mathbb{N}} \|C_n(\gamma)\| < \infty$ , for every  $\gamma \in \Gamma$ ;
- (3) for every  $C > 0$  there exists  $n_C \in \mathbb{N}$  such that for all  $n \geq n_C$  we have  $\{\gamma \in \Gamma_{n+1} \mid \|r_{n+1}(\gamma)\| \leq C\} \subset \Gamma_n$ .

For every  $\gamma \in \Gamma$  we define a map  $r : \Gamma \rightarrow \ell^2(\Gamma)$  by letting  $r(\gamma) = r_n(\gamma)$ , where  $n$  is chosen to be the smallest natural number such that  $\gamma \in \Gamma_n$ . The above properties then imply that  $r \in \mathcal{RA}(\Gamma, \cup_n \mathcal{G}_n, \ell^2(\Gamma))$ .

Doing some simple calculations the reader may verify that this above construction together with Proposition 1.10 in [9] shows that if there exists a sequence  $r_n \in \mathcal{RA}(\Gamma_n, \{e\}, \ell^2(\Gamma_n))$  with uniform bounded equivariance then  $r \in \mathcal{RA}(\oplus_n \Gamma_n, \{e\}, \ell^2(\oplus_n \Gamma_n))$ . In particular we have that if  $\mathcal{RA}(\Gamma, \{e\}, \ell^2(\Gamma)) \neq \emptyset$  then  $\mathcal{RA}(\Gamma^{\oplus \infty}, \{e\}, \ell^2(\Gamma^{\oplus \infty})) \neq \emptyset$ .

As expected, to obtain relative quasi-cocycles we have to impose stronger assumptions. For example, if there exist relative quasi-cocycles  $r_n \in \mathcal{RQ}(\Gamma_n, \mathcal{G}_n, \ell^2(\Gamma_n))$  satisfying  $\sup_{m,n} \sup_{\gamma \in \Gamma_{\min(n,m)}} \|r_n(\gamma) - r_m(\gamma)\| < \infty$ ,  $\sup_{n \in \mathbb{N}} D_n < \infty$ , and condition (3), the same construction as before shows that  $\mathcal{RQ}(\Gamma, \cup_n \mathcal{G}_n, \ell^2(\Gamma)) \neq \emptyset$ . Also we notice that by a basic rescaling procedure the same conclusion follows if we completely drop the uniform boundedness on the defects  $D_n$ , keep condition (3), and replace the first condition by the following: there exists a sequence  $K_n \geq D_n$  such that

$$\sup_{m,n} \sup_{\gamma \in \Gamma_{\min(n,m)}} \left\| \frac{1}{K_n} r_n(\gamma) - \frac{1}{K_m} r_m(\gamma) \right\| < \infty.$$

The examples presented above arise more or less from canonical algebraic constructions. More interestingly, relative quasi-cocycles on groups can be constructed naturally from purely geometric considerations. Below we single out a class of such examples which are intensely studied in geometric group theory.

$\rightsquigarrow$  **Relative hyperbolic groups.** The results in [31, 27, 28] imply that every Gromov-hyperbolic group  $\Gamma$  the pair  $(\Gamma, \{e\})$  admits a proper quasi-cocycle into a multiple of  $\ell^2(\Gamma)$  (Lemma 4.2 in [61]). Using a similar reasoning we will show a relative version of this result for the relatively hyperbolic groups in the sense of Bowditch [3].

Briefly, given a group  $\Gamma$  together with a family of subgroups  $\mathcal{G}$ , we say that  $\Gamma$  is hyperbolic relative to  $\mathcal{G}$  if there exists a graph  $\mathcal{K}$  on which  $\Gamma$  acts such that the following conditions are satisfied: a)  $\Gamma$  and every  $\Sigma \in \mathcal{G}$  are finitely generated, b)  $\mathcal{K}$  is fine (see (1) in Definition 2 from [3]) and has thin triangles, c) there are finitely many orbits and each edge stabilizer is finite, d) the infinite vertex stabilizers are precisely the elements of  $\mathcal{G}$  and their conjugates.

Here are some examples of relative hyperbolic groups: a free product is relative hyperbolic with respect to its factors; if  $\Gamma$  is hyperbolic relatively to a family of subgroups  $\mathcal{G}$  and  $\alpha : \Sigma_1 \rightarrow \Sigma_2$  is a monomorphism with  $\Sigma_i \in \mathcal{G}$ , then the HNN extension  $\Gamma \star_\alpha$  is hyperbolic with respect to  $\mathcal{G} \setminus \{\Sigma_1\}$  [11]; geometrically finite Kleinian

groups are hyperbolic with respect to their cusp subgroups; the fundamental group of a complete hyperbolic manifold of finite volume is hyperbolic relative to its cusp subgroups [13]; Sela's limit groups are hyperbolic relative to their maximal non-cyclic abelian subgroups [11].

In [29] Mineyev and Yaman showed that whenever  $\Gamma$  is hyperbolic relatively to a finite set  $\mathcal{G}$  of subgroups, there exists an ideal hyperbolic tuple  $(\Gamma, \mathcal{G}, X, \nu')$  (Definition 42 in [29]). Furthermore, using this in combination with the machinery developed in [27], they constructed a homological  $\mathbb{Q}$ -bicomplex in  $X$  which is  $\Gamma$ -equivariant, anti-symmetric, quasi-geodesic, and has bounded area (Theorem 47 in [29]). Therefore, applying the same arguments as in the proof of Theorem 7.13 of [32], we see that this bicomplex gives rise naturally to relative quasi-cocycles for  $(\Gamma, \mathcal{G})$  into a multiple of the left semi-regular representations of  $\Gamma$  with respect to some conjugates of elements in  $\mathcal{G}$ . Precisely, the bounded area together with anti-symmetry will imply the quasi-cocycle relation and being quasi-geodesic will imply properness with respect to the family  $\mathcal{G}$ .

**Proposition 2.5.** *If a group  $\Gamma$  is hyperbolic relative to a finite family of subgroups  $\mathcal{G}$  in the sense of Bowditch [3], then we have that  $\mathcal{RQ}(\Gamma, \mathcal{G}, \oplus_{i,j} \ell^2(\Gamma/\gamma_j \Sigma_i \gamma_j^{-1})) \neq \emptyset$ , for some  $g_j \in \Gamma$  and  $\Sigma_i \in \mathcal{G}$ .*

Another possible generalization for relative arrays is to assume that the elements of  $\mathcal{G}$  are just subsets of  $\Gamma$  rather than subgroups. This would enrich significantly the classes of examples presented before. However, in order to obtain structural von Neumann algebra results as in the sequel, further development of intertwining techniques would be required. It is more likely that this will be addressed on a case-by-case basis.

### 3. WEAK AMENABILITY FOR GROUPS AND VON NEUMANN ALGEBRAS

The notion of weak amenability for groups was introduced by Cowling and Haagerup in [10]. There are many equivalent definitions ([4, 10]) and for reader's convenience we recall the following:

**Definition 3.1.** A countable discrete group  $\Gamma$  is said to be weakly amenable with constant  $C$  if there exists a sequence of finitely supported functions  $\phi_n : \Gamma \rightarrow \mathbb{C}$  such that  $\phi_n \rightarrow 1$  pointwise and  $\limsup_n \|\hat{\phi}_n\|_{cb} \leq C$ , where  $\|\hat{\phi}\|_{cb}$  denotes the (completely bounded) norm of the Schur multiplier on  $\mathfrak{B}(\ell^2(\Gamma))$  associated with the kernel  $\hat{\phi}_n : \Gamma \times \Gamma \rightarrow \mathbb{C}$  given by  $\hat{\phi}_n(\gamma, \delta) = \phi(\gamma^{-1}\delta)$ .

The Cowling-Haagerup constant  $\Lambda_{cb}(\Gamma)$  is defined to be the infimum of all  $C$  for which such a sequence  $(\phi_n)$  exists. If  $\Gamma$  is not weakly amenable then we write  $\Lambda_{cb}(\Gamma) = \infty$ .

Below we summarize some families of groups known to be weakly amenable, also specifying their Cowling-Haagerup constants:

- (1) all amenable groups ( $\Lambda_{cb}(\Gamma) = 1$ );
- (2) all lattices in  $SO(n, 1)$  and  $SU(n, 1)$  ( $\Lambda_{cb}(\Gamma) = 1$ ) or lattices in  $Sp(n, 1)$  ( $\Lambda_{cb}(\Gamma) = 2n - 1$ ), [10];
- (3) Coxeter groups ( $\Lambda_{cb}(\Gamma) = 1$ ) [26];
- (4) more generally, all groups which act properly on finite dimensional CAT(0)-cube complexes ( $\Lambda_{cb}(\Gamma) = 1$ ), [16, 30];

- (5) all hyperbolic groups (in this case no explicit constants were computed), [37].
- (6) all limit groups in the sense of Sela ( $\Lambda_{cb}(\Gamma) = 1$ ); this is an observation due to Ozawa based on a result from [ ].

Groups which are not weakly amenable include  $\mathbb{Z}^2 \rtimes SL_2(\mathbb{Z})$ , [20] (see also, [12]), lattices in higher-rank Lie groups, and any non-amenable wreath products of the form  $\mathbb{Z} \wr \Sigma$ , [39].

The class of weakly amenable groups is closed under taking subgroups, cartesian products, co-amenable extensions, measure equivalence [39], and inductive limits of groups with uniformly bounded Cowling-Haagerup constants. However, it is not known whether weak amenability is closed under taking a free product of two groups except in the case that the Cowling-Haagerup constants of both groups are one [56].

By analogy with the group case discussed above, one can define a similar approximation property for von Neumann algebras. The precise formulation is the following

**Definition 3.2.** A von Neumann algebra  $M$  is said to have the *weak\* completely bounded approximation property*, abbreviated  $W^*CBAP$ , if there is a sequence of ultraweakly-continuous finite-rank maps  $(\phi_n)$  on  $M$  such that  $\phi_n \rightarrow \text{id}_M$  in the point-ultraweak topology and  $\limsup_n \|\phi_n\|_{cb} < \infty$ .

In [41] Ozawa and Popa discovered that the presence of this finite-dimensional approximation (with constant one) on a group imposes a certain type of “rigidity” on its internal structure. More precisely, they showed that if  $\Lambda_{cb}(\Gamma) = 1$  then for any amenable subgroup  $\Omega < \Gamma$  with non-amenable normalizing group  $N_\Gamma(\Omega)$  there exists an  $\Omega \rtimes N_\Gamma(\Omega)$  invariant state on  $\ell^\infty(\Omega)$ , where the semidirect product  $\Omega \rtimes N_\Gamma(\Omega)$  acts on  $\Omega$  by  $(\gamma, a) \cdot x = \gamma a x \gamma^{-1}$ . In other words, the natural action of the normalizer  $N_\Gamma(\Omega)$  on  $\Omega$  is fairly “small”; for instance, it cannot be of Bernoulli type. Later, Ozawa showed that in fact *all* weakly amenable groups satisfy this property, [39]. In fact, this rigidity even manifests in the von Neumann-algebraic context, as follows:

**Theorem 3.3** (Ozawa-Popa ’07, Ozawa ’10). *Let  $M$  be a von Neumann algebra which has  $W^*CBAP$  and let  $P \subset M$  be a diffuse amenable subalgebra. Then the natural action by conjugation of the normalizer  $\mathcal{N}_M(P) \curvearrowright P$  is weakly compact, i.e., there exists a net of unit vectors  $(\eta_n)_{n \in \mathbb{N}}$  in  $L^2(M) \bar{\otimes} L^2(\bar{M})$  such that:*

- (1)  $\|\eta_n - (v \otimes \bar{v})\eta_n\| \rightarrow 0$ , for all  $v \in \mathcal{U}(P)$ ;
- (2)  $\|[u \otimes \bar{u}, \eta_n]\| \rightarrow 0$ , for all  $u \in \mathcal{N}_M(P)$ ;
- (3)  $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n \rangle$ , for all  $x \in M$ .

In combination with deformation techniques, weak compactness turned out to be an powerful tool for obtaining many important structural results for group-measure space factors [41, 42, 18, 59].

#### 4. THE GAUSSIAN CONSTRUCTION, BIMODULES AND WEAK CONTAINMENT

Given an orthogonal representation  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H}_{\mathbb{R}})$  of a discrete, countable group there exists a way of associating to it a p.m.p. action of  $\Gamma$  on a measure space such that the induced Koopman representation equals  $\pi$ . This is called the Gaussian

construction associated to  $(\Gamma, \pi, \mathcal{H}_{\mathbb{R}})$ . We briefly describe this construction here, indicating how it can be extended to measure preserving actions by product groups.

If  $\pi : \Gamma \rightarrow \mathcal{O}(\mathcal{H}_{\mathbb{R}})$  is an orthogonal representation, the Gaussian construction as described in [45] or [59] provides a probability measure space  $Y_{\pi}$  and a family  $\omega(\xi)_{\xi \in \mathcal{H}}$  of unitaries in  $L^{\infty}(Y_{\pi})$  such that  $L^{\infty}(Y_{\pi})$  is generated as a von Neumann algebra by the  $\omega(\xi)$ 's and the following relations hold:

- (1)  $\omega(0) = 1$ ,  $\omega(\xi_1 + \xi_2) = \omega(\xi_1)\omega(\xi_2)$ ,  $\omega(\xi)^* = \omega(-\xi)$  for all  $\xi, \xi_1, \xi_2 \in \mathcal{H}_{\mathbb{R}}$
- (2)  $\tau(\omega(\xi)) = \exp(-\|\xi\|^2)$  where  $\tau$  is the trace on  $L^{\infty}(Y_{\pi})$  given by integration.

The action  $\sigma$  of  $\Gamma$  on  $L^{\infty}(Y_{\pi})$  is given by  $\sigma_g(\omega(\xi)) = \omega(\pi_g(\xi))$ , for all  $\xi \in \mathcal{H}_{\mathbb{R}}$ .

Suppose now that  $\Gamma_1 \times \Gamma_2$  acts trace preserving on an abelian von Neumann algebra  $(A, \tau)$  and denote by  $M = A \rtimes (\Gamma_1 \times \Gamma_2)$  the corresponding crossed product von Neumann algebra.

For each  $i = 1, 2$  let  $\pi_i : \Gamma_i \rightarrow \mathcal{O}(\mathcal{H}_i)$  be an orthogonal representation which is weakly contained in the (real) left regular representation of  $\Gamma_i$  and let  $L^2(Y_{\pi_i})_0 = L^2(Y_{\pi_i}) \ominus \mathbb{C}1$  be the Koopman representation of the Gaussian action corresponding to  $\pi_i$ . Consider the Hilbert space  $\mathcal{K} = L^2(A) \bar{\otimes} L^2(Y_{\pi_1})_0 \bar{\otimes} L^2(Y_{\pi_2})_0 \bar{\otimes} \ell^2(\Gamma_1 \times \Gamma_2)$  with the  $M$ -bimodular structure defined as

$$(au_g) \cdot (\xi \otimes \xi_1 \otimes \xi_2 \otimes \delta_k) \cdot (bu_h) = (a\sigma_g(\xi)\sigma_{gk}(b)) \otimes (\pi_g(\xi_1 \otimes \xi_2)) \otimes (\delta_{gkh}),$$

for every  $a, b \in A$ ,  $\xi \in L^2(A)$ ,  $\xi_1 \in L^2(Y_{\pi_1})_0$ ,  $\xi_2 \in L^2(Y_{\pi_2})_0$ , and  $g, k, h \in \Gamma_1 \times \Gamma_2$ . Here  $\pi = \pi_1 \otimes \pi_2$ .

One of the key ingredients needed in the proof of Theorem 6.1 is that whenever  $A$  is amenable the above bimodule be weakly contained in the coarse  $M$ -bimodule. The following lemma takes care of this.

**Lemma 4.1** (Fell's absorption principle). *As an  $M$ -bimodule,  $\mathcal{K}$  is isomorphic with a multiple of  $L^2(\langle M, A \rangle, Tr)$ . In particular, when  $A$  is amenable, it follows that  $\mathcal{K}$  is weakly contained in the coarse bimodule,  $L^2(M) \bar{\otimes} L^2(M)$ .*

*Proof.* First we notice that when  $\pi_i$  is weakly contained in  $\rho_i$  then the bimodule associated to the pair  $(\pi_1, \pi_2)$  is weakly contained in the bimodule associated with the pair  $(\rho_1, \rho_2)$ . It is therefore enough to prove the statement in the case when  $\pi_i$  is the (real) left representations of  $\Gamma_i$ .

For simplicity, throughout this proof, we denote by  $\Gamma = \Gamma_1 \times \Gamma_2$ . Since  $\mathcal{K}$  is canonically identified with  $L^2(A) \bar{\otimes} \ell^2(\Gamma) \bar{\otimes} \ell^2(\Gamma)$ , we will obtain the desired conclusion by showing that the map

$$L^2(A) \bar{\otimes} \ell^2(\Gamma) \bar{\otimes} \ell^2(\Gamma) \ni \xi \otimes \delta_g \otimes \delta_h \rightarrow \xi u_g e_A u_{g^{-1}h} \in L^2(\langle M, A \rangle, Tr)$$

implements an isomorphism between the two bimodules.

To this purpose it suffices to show that

$$(3) \quad \|(au_s) \cdot (\xi \otimes \delta_g \otimes \delta_h) \cdot (bu_t)\|_2^2 = \|au_s(\xi u_g e_A u_{g^{-1}h}) bu_t\|_{2, Tr}^2,$$

for all  $a, b \in A$ ,  $\xi \in L^2(A)$ , and  $s, t, g, h \in \Gamma$ .

On one hand, by definitions, the left side in the previous equation is equal to

$$\begin{aligned} \|(au_s) \cdot (\xi \otimes \delta_g \otimes \delta_h) \cdot (bu_t)\|_2^2 &= \|(a\sigma_s(\xi)\sigma_{sh}(b)) \otimes \delta_{sg} \otimes \delta_{sh}\|_2^2 \\ &= \|(a\sigma_s(\xi)\sigma_{sh}(b))\|_2^2 \\ &= \|au_s \xi u_h b\|_2^2. \end{aligned}$$

On the other hand, since  $u_{g^{-1}h} b b^* u_{h^{-1}g}$  belongs to  $A$  (hence, commutes with  $e_A$ ) we see that the right side of (3) is equal to

$$\begin{aligned}
\|au_s(\xi u_g e_A u_{g^{-1}h})bu_t\|_{2,Tr}^2 &= Tr(b^* u_{h^{-1}g} e_A u_{g^{-1}} \xi^* u_{s^{-1}} a^* a u_s \xi u_g e_A u_{g^{-1}h} b) \\
&= Tr(u_{g^{-1}} \xi^* u_{s^{-1}} a^* a u_s \xi u_g u_{g^{-1}h} b b^* u_{h^{-1}g} e_A) \\
&= \tau(u_{g^{-1}} \xi^* u_{s^{-1}} a^* a u_s \xi u_g u_{g^{-1}h} b b^* u_{h^{-1}g}) \\
&= \tau(\xi^* u_{s^{-1}} a^* a u_s \xi u_h b b^* u_{h^{-1}}) \\
&= \|au_s \xi u_h b\|_2^2.
\end{aligned}$$

This establishes (3) and hence the conclusion of the lemma.  $\square$

## 5. A PATH OF AUTOMORPHISMS OF THE EXTENDED ROE ALGEBRA ASSOCIATED WITH THE PRODUCTS OF GAUSSIAN ACTIONS

Let  $\Gamma = \Gamma_1 \times \Gamma_2 \curvearrowright X$  be a measure preserving action of  $\Gamma$  on a measure space  $X$ . Assume we are given orthogonal representations  $\pi_i : \Gamma_i \rightarrow \mathcal{O}(\mathcal{H}_i)$ . As shown in the previous section, to these representations we can associate the Gaussian actions  $\Gamma_i \curvearrowright^{\pi_i} (Y_{\pi_i}, \nu_i)$  (in a slight abuse of notation we will denote the Gaussian action by the same letter). Next we consider the product action  $\Gamma \curvearrowright^{\pi_1 \otimes \pi_2} (Y_{\pi_1} \times Y_{\pi_2}, \nu_1 \times \nu_2)$  and the diagonal action of  $\Gamma$  on  $(X \times Y_{\pi_1} \times Y_{\pi_2}, \mu \times \nu_1 \times \nu_2)$ . To this action, following [9], we can associate the extended Roe algebra  $C_u^*(\Gamma \curvearrowright Z)$  (where the action is the one above and  $Z = X \times Y_{\pi_1} \times Y_{\pi_2}$ ).

Additionally, given any pair of quasi-cocycles  $q_i : \Gamma_i \rightarrow \mathcal{H}_i$  for the respective representations  $\pi_i$ , we can construct a one-parameter family  $(\alpha_t)_{t \in \mathbb{R}}$  of  $*$ -automorphisms of  $C_u^*(\Gamma \curvearrowright Z)$ , by exponentiating the  $q_i$ 's. This traces back to the construction of a malleable deformation of  $L\Gamma$  from a cocycle  $b$  as carried out in §3 of [59]. Moreover, this family will be pointwise continuous with respect to the uniform norm as  $t \rightarrow 0$  (Theorem 5.3).

Given the quasi-cocycles  $q_i : \Gamma_i \rightarrow \mathcal{H}_i$ , one can construct, following section §1.2 of [59], two one-parameter families of maps  $v_t^i : \Gamma_i \rightarrow \mathcal{U}(L^\infty(Y_{\pi_i}, \nu_i))$  defined by the formula  $v_t^i(\gamma_i)(x) = \exp(\sqrt{-1}t q_i(\gamma_i)(x))$ , where  $\gamma_i \in \Gamma_i$ ,  $x \in Y_{\pi_i}$ , respectively. To understand this formula, the reader must think about  $\mathcal{H}_i$  as being identified with  $L^\infty(Y_{\pi_i}, \nu_i)$ , viewing the elements  $q_i(\gamma_i)$  as functions on  $Y_{\pi_i}$ . The same computations as in [45, 59] show the following:

**Proposition 5.1.** *Assuming the same notations as above, we have that:*

- (1) *If the representation  $\pi_i$  is weakly- $\ell^2$  for every  $i = 1, 2$  then the (tensor) product of Koopman representations  $\pi_1 \otimes \pi_2|_{L_0^2(Y_{\pi_1}) \otimes L_0^2(Y_{\pi_2})}$  is also weakly- $\ell^2$ ;*
- (2)  *$\int_Y^{\pi_i} v_t^i(\gamma_i)(y) v_t^i(\delta_i)^*(y) d\mu^{\pi_i}(y) = \kappa_t^i(\gamma_i, \delta_i)$  for all  $i = 1, 2$  and  $\gamma_i, \delta_i \in \Gamma_i$ .*

Here,  $\kappa_t^i(\gamma_i, \delta_i) = \exp(-t \|q_i(\gamma_i) - q_i(\delta_i)\|)$ .

With the help of these maps we can construct a path of unitary operators  $V_t \in \mathfrak{B}(L^2(Z) \bar{\otimes} \ell^2(\Gamma)) = \mathfrak{B}(L^2(Y_{\pi_1}) \bar{\otimes} L^2(Y_{\pi_2}) \bar{\otimes} L^2(X) \bar{\otimes} \ell^2(\Gamma))$  by letting  $V_t(\xi_1 \otimes \xi_2 \otimes \eta \otimes \delta_{(\gamma_1, \gamma_2)}) = v_t^1(\gamma_1) \xi_1 \otimes v_t^2(\gamma_2) \xi_2 \otimes \eta \otimes \delta_{(\gamma_1, \gamma_2)}$  for every  $\eta \in L^2(X)$ ,  $\xi_i \in L^2(Y_{\pi_i})$ , and  $\gamma_i \in \Gamma_i$ , where  $i = 1, 2$ . The computations in [9] show that the  $V_t$ 's enjoy the following basic properties of  $V_t$ .

**Proposition 5.2.** *For every  $t, s \in \mathbb{R}$  we have that:*

- (1)  $V_t V_s = V_{t+s}$ ,  $V_t V_t^* = V_t^* V_t = 1$

- (2)  $JV_tJ = V_t$  where  $J : L^2(Z) \bar{\otimes} \ell^2(\Gamma) \rightarrow L^2(Z) \bar{\otimes} \ell^2(\Gamma)$  is Tomita's conjugation.

The unitary  $V_t$  implements an inner  $\star$ -automorphism  $\alpha_t$  on  $\mathfrak{B}(L^2(Z) \bar{\otimes} \ell^2(\Gamma))$  by letting  $\alpha_t(x) = V_t x V_t^*$  for all  $x \in \mathfrak{B}(L^2(Z) \bar{\otimes} \ell^2(\Gamma))$ . The  $\alpha_t$  then restricts to a family of inner automorphisms of the uniform Roe algebra. Moreover, when restricting to the uniform Roe algebra one can recover from  $\alpha_t$  the multipliers introduced above:  $E_M \circ \alpha_t(x) = \mathbf{m}_t(x)$  for all  $x \in C_u^*(\Gamma)$ . Same computations as in [9] can be used to show that, when properly restricted,  $\alpha_t$  is  $C^*$ -deformation i.e., pointwise- $\|\cdot\|_\infty$  continuous.

**Theorem 5.3.** *Assuming the notations above, for every  $x \in L^\infty(X) \rtimes_{\sigma,r} \Gamma$  we have*

- (4)  $\|(\alpha_t(x) - x) \circ e\|_\infty \rightarrow 0$  as  $t \rightarrow 0$ ;  
 (5)  $\|(\alpha_t(JxJ) - JxJ) \circ e\|_\infty \rightarrow 0$  as  $t \rightarrow 0$ .

where  $\|\cdot\|_\infty$  denotes the operatorial norm in  $\mathfrak{B}(L^2(X) \bar{\otimes} \ell^2(\Gamma))$ . Here  $e$  denotes the orthogonal projection from  $L^2(Z) \bar{\otimes} \ell^2(\Gamma)$  onto  $L^2(X) \bar{\otimes} \ell^2(\Gamma)$ .

## 6. PROOFS OF THE MAIN RESULTS INVOLVING PRODUCT OF GROUPS

We start by proving the main technical result of the paper which involves product of groups. Specifically, we obtain a result describing all weakly compact embeddings in the crossed product von Neumann algebras arising from actions of products of hyperbolic groups (Theorem 6.1). Our approach follows the general outline of the proof of Theorem B in [42] and Theorem B in [9]. However, it is based on a new ingredient which allows us to treat the more general case of arrays rather than just quasi-cocycles as proved in [9]. This was influenced by the approach taken in [7].

**Theorem 6.1.** *For  $i = 1, 2$  let  $\Gamma_i$  be an exact group with a quasi-normal subgroup  $\Sigma_i < \Gamma_i$  and let  $\pi_i : \Gamma_i \rightarrow \mathcal{U}(\mathcal{H}_{\pi_i})$  be a weakly- $\ell^2$  representation such that  $\mathcal{RA}(\Gamma_i, \{\Sigma_i\}, \mathcal{H}_{\pi_i}) \neq \emptyset$ . Let  $\Gamma_1 \times \Gamma_2 \curvearrowright X$  be a measure-preserving action on a probability space and denote by  $M = L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$ . If  $P \subset M$  is a weakly compact embedding (cf. [41]), then one can find projections  $p_0, p_1, p_2, p_3 \in \mathcal{Z}(\mathcal{N}_M(P)' \cap M)$  with  $p_0 + p_1 + p_2 + p_3 = 1$  such that the following hold:*

- (1)  $\mathcal{N}_M(P)'' p_0$  is amenable;
- (2)  $P p_1 \preceq_M L^\infty(X) \rtimes (\Gamma_1 \times \Sigma_2)$ ;
- (3)  $P p_2 \preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Gamma_2)$ ;
- (4)  $P p_3 \preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$ .

*Proof.* For simplicity we establish the following notations:  $M_1 = L^\infty(X) \rtimes (\Gamma_1 \times \Sigma_2)$ ,  $M_2 = L^\infty(X) \rtimes (\Sigma_1 \times \Gamma_2)$ ,  $N = \mathcal{N}_M(P)''$  and  $\mathcal{Z} = \mathcal{Z}(N' \cap M)$ . Let  $p_0 \in \mathcal{Z}$  be the maximal projection such that  $N p_0$  is amenable. Let  $p_1 \in (P' \cap M)(1 - p_0)$  be a maximal projection satisfying (2) together with  $P q_1 \not\preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$  for all  $0 \neq q_1 \leq p_1$  (obtained via a standard maximality argument). Similarly let  $p_2 \in (P' \cap M)(1 - p_0)$  be a maximal projection satisfying (3) together with  $P q_2 \not\preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$  for all  $0 \neq q_2 \leq p_2$ . Also we denote by  $p_3 \in (P' \cap M)(1 - p_0)$  be a maximal projection satisfying (4). By maximality, we must have that  $p_1, p_2, p_3 \in \mathcal{Z}(P' \cap M)$ . Moreover, we have that  $p_1, p_2, p_3 \in \mathcal{Z}$ . Indeed, if  $u \in \mathcal{N}_M(P)$ , let  $\tilde{p}_3 = u p_3 u^*(p_0 - p_3)$ . Then  $p_3 + \tilde{p}_3$  also satisfies (4) and by the maximality of  $p_3$ , we get that  $\tilde{p}_3 = 0$ . Thus  $p_3 = u p_3 u^*$ , for every  $u \in \mathcal{U}(P)$ , hence  $p_3 \in \mathcal{Z}$ .

Notice that from the definitions we have that  $p_1 p_3 = p_2 p_3 = 0$ . Further, we claim that  $p'_1 := p_1 p_2 = 0$ . Otherwise,  $Pp'_1 \preceq_M M_1$ , hence we can find projections  $p \in P$ ,  $p' \in P' \cap M$ , a non-zero partial isometry  $v$  and a  $*$ -homomorphism  $\psi : pPpp'_1 \rightarrow M_1$  such that  $v^*v = pp'p'_1$  and  $\psi(x)v = vx$ , for all  $x \in pPpp'_1$ . Let  $z$  denote the central support of  $p$  in  $P$  and set  $p'_2 = zp'p'_1 \in P' \cap M$ . By using the fact that  $pPpp'_1 \subset v^*M_1v$ , for every  $\varepsilon > 0$ , we can find a finite set  $W \subset \Gamma_2$  such that  $\|x - \sum_{g \in W} E_{M_1}(xu_g^*)u_g\|_2 \leq \varepsilon$ , for all  $x \in Pp'_2$  with  $\|x\| \leq 1$ .

Since  $p'_2 \leq p_2$  then  $Pp'_2 \not\preceq_M L^\infty(X) \rtimes (\Sigma_1 \times \Sigma_2)$  and combining this with the above it follows that  $Pp'_2 \not\preceq_M M_2$ , contradicting the fact that  $0 \neq p'_2 \leq p_2$ .

Therefore, to prove the theorem, we only need to show that  $p_0 + p_1 + p_2 + p_3 = 1$ . By contradiction, assume that  $p := 1 - (p_0 + p_1 + p_2 + p_3) \neq 0$ . Note that  $Pp \not\preceq_M M_1$  and  $Pp \not\preceq_M M_2$ . Indeed, if  $Pp \preceq_M M_1$ , then by reasoning as in the previous paragraph we can find a non-zero projection  $\tilde{p}_1 \in (P' \cap M)p$  such that  $P\tilde{p}_1 \preceq_M^s M_1$ . Thus,  $P(p_1 + \tilde{p}_1) \preceq_M^s M_1$ , which contradicts the maximality of  $p_1$ . Also, note that  $Np$  has no amenable direct summand. If  $N\tilde{p}_0$  is amenable for some non-zero projection  $\tilde{p}_0 \in \mathcal{Z}p$ , it follows that  $N(p_0 + \tilde{p}_0)$  is also amenable, contradicting the maximality of  $p_0$ .

By assumption  $P \subset M$  is weakly compact, so there exists a net of unit vectors  $(\eta_n)_{n \in \mathbf{N}}$  in  $L^2(M) \bar{\otimes} L^2(\bar{M})$  such that:

- (1)  $\|\eta_n - (v \otimes \bar{v})\eta_n\| \rightarrow 0$ , for all  $v \in \mathcal{U}(P)$ ;
- (2)  $\|[u \otimes \bar{u}, \eta_n]\| \rightarrow 0$ , for all  $u \in \mathcal{N}_M(P)$ ;
- (3)  $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n \rangle$ , for all  $x \in M$ .

As in the previous section we consider the von Neumann algebra

$$\tilde{M} = (L^\infty(X) \bar{\otimes} L^\infty(Y_{\pi_1}) \bar{\otimes} L^\infty(Y_{\pi_2})) \rtimes (\Gamma_1 \times \Gamma_2),$$

together with the following subalgebras:

$$\begin{aligned} \tilde{M}_1 &= (L^\infty(X) \bar{\otimes} L^\infty(Y_{\pi_1}) \bar{\otimes} \mathbb{C}1) \rtimes (\Gamma_1 \times \Gamma_2) \\ \tilde{M}_2 &= (L^\infty(X) \bar{\otimes} \mathbb{C}1 \bar{\otimes} L^\infty(Y_{\pi_2})) \rtimes (\Gamma_1 \times \Gamma_2) \end{aligned}$$

For every  $1 \leq i \leq 2$  we denote by  $e_i$  the orthogonal projection from  $L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M})$  onto  $L^2(\tilde{M}_i) \bar{\otimes} L^2(\bar{M})$  and notice that  $e_1$  and  $e_2$  commutes. It follows that  $e = (1 - e_1)(1 - e_2)$  is a projection and we denote its image by  $\mathcal{K} = e \left( L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M}) \right)$ .

Since  $e_i$  is  $M$ -bimodular we have that  $\mathcal{K}$  is an  $M \bar{\otimes} \bar{M}$ -subbimodule of  $L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M})$ .

Fixing  $t > 0$  we consider the unitary  $V_t$  associated with  $q_i \in \mathcal{RA}(\Gamma_i, \{\Sigma_i\}, \mathcal{H}_{\pi_i})$  as defined in the previous section and we then denote by

$$\tilde{\eta}_{n,t} = (V_t \otimes 1)(p \otimes 1)\eta_n \in L^2(\tilde{M}) \bar{\otimes} L^2(\bar{M}).$$

Using these notations we show next the following inequality:

**Lemma 6.2.**

$$\lim_n \|e(\tilde{\eta}_{n,t})\| \geq \frac{1}{2} \|p\|_2.$$

*Proof.* We argue by contradiction, so passing to a subsequence we assume that

$$(2) \quad \|e(\tilde{\eta}_{n,t})\| < \frac{1}{2} \|p\|_2 \text{ for all } n.$$



Denoting by  $\zeta_n = (p \otimes 1)\eta_n$  we observe that  $\|\tilde{\eta}_{n,t}\|_2 = \|\zeta_n\|_2 = \|p\|_2$ . Since by construction  $e + (e_1 - e_1 e_2) + e_2 = 1$ , we get that

$$(7) \quad \|e(\tilde{\eta}_{n,t})\|_2^2 + \|e_1(\tilde{\eta}_{n,t})\|_2^2 + \|e_2(\tilde{\eta}_{n,t})\|_2^2 \geq \|\tilde{\eta}_{n,t}\|_2^2 = \|p\|_2^2.$$

Since we assumed that  $\|e(\tilde{\eta}_{n,t})\|_2 \leq \frac{1}{2}\|p\|_2$ , for all  $n \geq n_0$ , after passing to a subsequence and without any loss of generality, we may assume that

$$(8) \quad \|e_1(\tilde{\eta}_{n,t})\|_2 \geq \sqrt{\frac{3}{8}}\|p\|_2 \text{ for all } n.$$

Throughout the proof, for any subset  $\mathcal{F} \subset \Gamma_1$ , we denote by  $P_{\mathcal{F}}$  the orthogonal projection from  $L^2(M) \bar{\otimes} L^2(\bar{M})$  onto the closed linear span of the set  $\{(M_2)u_h \bar{\otimes} \bar{M} \mid h \in \mathcal{F}\}$ .

The main strategy is to prove that relation (8) together with the assumption  $Pp \not\leq_M M_2$  will enable us to construct an infinite sequence  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots$  of finite subsets of  $\Gamma$  such that  $\Sigma_1 \mathcal{F}_i$  are disjoint and which satisfies the following property: for every  $k \in \mathbb{N}$  we can find  $n_k \in \mathbb{N}$  such that for all  $i \leq k$  and  $n \geq n_k$  we have

$$(9) \quad \|P_{\mathcal{F}_i}(\zeta_n)\| \geq \frac{1}{10}\|p\|_2.$$

First we briefly explain how this claim leads to a contradiction, thus finishing the proof of the lemma. Since the sets  $\Sigma_1 \mathcal{F}_i$  are disjoint, relation (9) implies

$$\|p\|_2^2 = \|\zeta_n\|_2^2 \geq \sum_{i=1}^k \|P_{\mathcal{F}_i}(\zeta_n)\|_2^2 \geq k \left( \frac{1}{10}\|p\|_2 \right)^2, \text{ for all } k \in \mathbb{N} \text{ and } n \geq n_k. \text{ This is obviously impossible when letting } k \text{ be sufficiently large.}$$

So we are left to prove (9). To show this we will proceed by induction on  $k$ .

First we prove case  $k = 1$ . Since  $\zeta_n \in L^2(M) \bar{\otimes} L^2(\bar{M})$ , we write  $\zeta_n = \sum_{g \in \Gamma_1} \zeta_g^n \delta_g$ , where  $\zeta_g^n \in L^2(X) \rtimes (1 \times \Gamma_2) \bar{\otimes} L^2(\bar{M})$ . Then, using the definition of  $V_t$ , a straight forward computation shows that

$$\|e_1(\tilde{\eta}_{n,t})\|^2 = \sum_{g \in \Gamma_1} \exp(-2t^2\|q_1(g)\|^2) \|\zeta_g^n\|^2, \text{ for all } n.$$

When combined with (8) this formula implies that, for all  $n$  we have

$$(10) \quad \sum_{g \in \Gamma_1} \exp(-2t^2\|q_1(g)\|^2) \|\zeta_g^n\|^2 > \frac{3}{8}\|p\|_2^2.$$

Since the map  $\Gamma_1 g \rightarrow \|q_1(g)\|$  is a proper relatively to  $\{\Sigma_1\}$  and  $\Sigma_1$  is quasi-normal in  $\Gamma_1$ , then the set  $\{g \in \Gamma_1 \mid \exp(-t^2\|q_1(g)\|^2) \geq \frac{1}{4}\}$  is contained in  $\Sigma_1 \mathcal{F}$  for some finite set  $\mathcal{F} \subset \Gamma$  and, using the inequality (10), we further deduce that

$$\frac{3}{8}\|p\|_2^2 < \frac{1}{16} \sum_{g \in \Gamma_1 \setminus \Sigma_1 \mathcal{F}} \|\zeta_g^n\|^2 + \sum_{g \in \Sigma_1 \mathcal{F}} \|\zeta_g^n\|^2, \text{ for all } n.$$

By basic algebraic manipulations, the above inequality gives that  $\sum_{g \in \Sigma_1 \mathcal{F}} \|\zeta_g^n\|^2 >$

$\frac{1}{3}\|p\|_2^2$  for all  $n$  which implies

$$(11) \quad \|P_{\mathcal{F}}(\zeta_n)\| > \frac{1}{\sqrt{3}}\|p\|_2.$$

So case  $k = 1$  follows by letting  $\mathcal{F}_1 = \mathcal{F}$  and  $n_1 = 1$ .

Next we show the induction step, i.e., assuming that we have constructed the sets  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k \subset \Gamma$  and  $n_k \in \mathbb{N}$ , we indicate how to construct  $\mathcal{F}_{k+1} \subset \Gamma$  and  $n_{k+1} \in \mathbb{N}$  satisfying (9).

Consider the set  $\mathcal{G} = \cup_{i=1}^k (\mathcal{F}_i \mathcal{F}^{-1}) \subset \Gamma$ . Since  $\mathcal{G}_1$  is finite and  $Pp \not\leq_M M_2$ , by Popa's intertwining techniques, there exist a unitary  $v \in \mathcal{U}(P)$ , a finite set  $\mathcal{K} \subset \Gamma_1$ , and an element  $v'$  in the linear span of  $\{M_2 u_h \mid h \in \mathcal{K}\}$  such that

$$(12) \quad \Sigma_1 \mathcal{K} \cap \Sigma_1 \mathcal{G} = \emptyset;$$

$$(13) \quad \|v' - vp\|_2 \leq \frac{1}{40|\mathcal{F}|} \|p\|_2.$$

Next, we show that for  $n \in \mathbb{N}$  and  $z \in M$  we have

$$(14) \quad \|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq |\mathcal{F}| \|z\|_2.$$

Fix  $n$  and denote by  $P$  the orthogonal projection onto  $L^2(M_2) \bar{\otimes} L^2(\bar{M})$ , i.e.  $P = P_{\emptyset}$ . We have  $P_{\mathcal{F}}(\zeta_n) = \sum_{h \in \mathcal{F}} P(\zeta_n(u_h^* \otimes 1))(u_h \otimes 1)$  and by the Cauchy-Schwarz inequality we deduce

$$(15) \quad \|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\|^2 \leq |\mathcal{F}| \sum_{h \in \mathcal{F}} \|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2$$

Now we let  $E_{M_2}$  to be the conditional expectation from  $M$  onto  $M_2$  and we denote by  $a = E_{M_2}(z^* z)^{\frac{1}{2}}$ . Using the formulas  $\langle (x \otimes 1)P(\zeta), P(\zeta) \rangle = \langle (E_{M_2}(x) \otimes 1)P(\zeta), P(\zeta) \rangle$  and  $\|(x \otimes 1)\eta_n\| = \|x\|_2$ , for all  $\zeta \in L^2(M) \bar{\otimes} L^2(\bar{M})$  and  $x \in M$ , we obtain the following:

$$\begin{aligned} & \|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2 \\ &= \langle (z^* z \otimes 1)P(\zeta_n(u_h^* \otimes 1)), P(\zeta_n(u_h^* \otimes 1)) \rangle \\ &= \langle (E_{M_2}(z^* z) \otimes 1)P(\zeta_n(u_h^* \otimes 1)), P(\zeta_n(u_h^* \otimes 1)) \rangle \\ &= \|(a \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|_2^2 = \|P((a \otimes 1)\zeta_n(u_h^* \otimes 1))\|^2 \\ &\leq \|(a \otimes 1)\zeta_n\|^2 = \|ap\|_2^2 \leq \|a\|_2^2 = \|z\|_2^2. \end{aligned}$$

It is clear that the last inequalities combined with (15) give (14).

To this end, applying the triangle inequality, for all  $v \in \mathcal{U}(P)$  and all  $n \in \mathbb{N}$ , we have

$$(16) \quad \begin{aligned} & \|\zeta_n - (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \\ & \|\zeta_n - (vp \otimes \bar{v})\zeta_n\| + \|\zeta_n - P_{\mathcal{F}}(\zeta_n)\| + \|((v' - vp) \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \end{aligned}$$

Since  $p$  and  $v$  commute, we have  $\zeta_n - (vp \otimes \bar{v})\zeta_n = (p \otimes 1)(\eta_n - (v \otimes \bar{v})\eta_n)$ . Thus, since  $\lim_{n \rightarrow \infty} \|\eta_n - (v \otimes \bar{v})\eta_n\|_2 = 0$ , we can find  $n_{k+1} \geq n_k$  such that for all  $n \geq n_{k+1}$  we have

$$(17) \quad \|\zeta_n - (vp \otimes \bar{v})\zeta_n\| \leq \frac{1}{40} \|p\|_2.$$

Using (14) for  $z = vp - v'$  in combination with (13) for all  $n$  we have

$$(18) \quad \|((v' - vp) \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{40} \|p\|_2.$$

Altogether, (16), (17), (18), and (30) show that for all  $n \geq n_{k+1}$  we have

$$(19) \quad \|\zeta_n - (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{20}\|p\|_2 + \|(\zeta_n - P_{\mathcal{F}}(\zeta_n))\| < \frac{3 + 20\sqrt{6}}{60}\|p\|_2.$$

Finally we let  $\mathcal{F}_{k+1} = \mathcal{K}\mathcal{F}$  and by (12) we see that  $\Sigma_1\mathcal{F}_{k+1}$  is disjoint from  $\Sigma_1\mathcal{F}, \Sigma_1\mathcal{F}_2, \dots, \Sigma_1\mathcal{F}_k$ . Moreover, we have that  $(v' \otimes v)P_{\mathcal{F}}(\zeta_n)$  belongs to the closed linear span of  $\{(M_2u_h) \otimes \bar{M} \mid h \in \mathcal{F}_{k+1}\}$ . Thus,  $P_{\mathcal{F}_{k+1}}((v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)) = (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)$  and (19) together with the triangle inequality give that

$$\|\zeta_n - P_{\mathcal{F}_{k+1}}(\zeta_n)\| < \frac{3 + 20\sqrt{6}}{60}\|p\|_2, \text{ for all } n \geq n_{k+1}.$$

Hence, for all  $n \geq n_{k+1}$ , we have

$$\|P_{\mathcal{F}_{k+1}}(\zeta_n)\| > \left(1 - \left(\frac{3 + 20\sqrt{6}}{60}\right)^2\right)^{\frac{1}{2}} \|p\|_2 > \frac{1}{10}\|p\|_2,$$

which ends the proof of (9).  $\square$

We also notice that from Lemma 4.1 it follows that, as an  $M$ -bimodule,  $\mathcal{K}$  is weakly contained in the coarse bimodule. Following the same argument as in Theorem B of [42] we define a state  $\psi_t$  on  $\mathcal{N} = \mathfrak{B}(\mathcal{K}) \cap \rho(M^{op})'$ . Explicitly, if we denote by  $\xi_{n,t} = e(\tilde{\eta}_{n,t})$  we let  $\psi_t(x) = \text{Lim}_n \frac{1}{\|\xi_{n,t}\|_2} \langle (x \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle$  for every  $x \in \mathcal{N}$ . Next we recall that from [9] we have the following two lemmas

**Lemma 6.3** (Lemma 4.3 in [9]). *For every  $\varepsilon > 0$  and every finite set  $K \subset L^\infty(X) \rtimes_{\sigma,r} (\Gamma_1 \times \Gamma_2)$  with  $\text{dist}_{\|\cdot\|_2}(y, (N)_1) \leq \varepsilon$  for all  $y \in K$  one can find  $t_\varepsilon > 0$  and a finite set  $L_{K,\varepsilon} \subset \mathcal{N}_M(P)$  such that*

$$(20) \quad |\langle (yx - xy) \otimes 1 \rangle \xi_{n,t}, \xi_{n,t} \rangle| \leq 10\varepsilon + 2 \sum_{v \in L_{K,\varepsilon}} \|[v \otimes \bar{v}, \eta_n]\|,$$

for all  $y \in K$ ,  $\|x\|_\infty \leq 1$ ,  $t_\varepsilon > t > 0$ , and  $n$ .

**Lemma 6.4** (Lemma 4.4 in [9]). *For every  $\varepsilon > 0$  and any finite set  $F_0 \subset \mathcal{U}(N)$  there exist a finite set  $F_0 \subset F \subset M$ , a c.c.p. map  $\varphi_{F,\varepsilon} : \text{span}(F) \rightarrow L^\infty(X) \rtimes_{\sigma,r} (\Gamma_1 \times \Gamma_2)$ , and  $t_\varepsilon > 0$  such that*

$$(21) \quad |\psi_{t_\varepsilon}(\varphi_{F,\varepsilon}(up)^* x \varphi_{F,\varepsilon}(up)) - \psi_{t_\varepsilon}(x)| \leq 116\varepsilon,$$

for all  $u \in F_0$  and  $\|x\|_\infty \leq 1$ .

For the remaining part of the proof we mention that one can use Haagerup criterion to show that  $Np$  is amenable. In fact the reasoning in Theorem B in [42] applies verbatim in our case and we leave the details to the reader.  $\square$

We notice that even though we choose to state the result only for one quasi-normal subgroup, the same method can be used to treat the case of arbitrary families of quasi-normal subgroups. This proof can be upgraded to work for any families of subgroups [54].

*Proof of Corollary 0.4.* Applying the previous theorem for  $A = \mathbb{C}1$  and  $\Sigma_i = e$  there exist  $p_0, p_1, p_2 \in \mathcal{Z}$  with  $p_0 + p_1 + p_2 = 1$  such that  $p_0\mathcal{N}_M(P)''$  is amenable,  $p_1B \preceq_M L\Gamma_1$ , and  $p_2B \preceq_M L\Gamma_2$ . Therefore the conclusion follows if we show that  $p_0 = 1$ . Assuming this is not the case one can find  $p_1 \neq 0$  such that  $p_1B \preceq_M L\Gamma_1$ . Then Remark 3.8 in [63] implies that  $L\Gamma'_1 \cap M \preceq_M p_1B' \cap M$  and since  $L\Gamma'_1 \cap M =$

$L\Gamma_2$  then we have  $L\Gamma_2 \preceq_M p_1 B' \cap M$ . This however is a contradiction because  $L\Gamma_2$  is a non-amenable factor while  $p_1 B' \cap M$  is assumed to be an amenable algebra.  $\square$

*Proof of Corollary 0.5.* Assume that  $\Lambda \curvearrowright Y$  is a free, ergodic action which is  $W^*$ -equivalent to  $\Gamma_1 \times \Gamma_2 \curvearrowright X$ . This amounts to the existence of an  $\star$ -isomorphism  $\psi : L^\infty(Y) \rtimes \Lambda \rightarrow L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$ . For simplicity we will denote by  $A = L^\infty(X)$ ,  $B = L^\infty(Y)$ ,  $M = A \rtimes (\Gamma_1 \times \Gamma_2)$ ,  $M_1 = A \rtimes \Gamma_1$ , and  $M_2 = A \rtimes \Gamma_2$ .

Below we will prove that there exists a unitary  $x \in \mathcal{U}(M)$  such that  $x\psi(B)x^* = A$ . Notice that since  $C = \psi(B)$  Cartan in  $M$  its normalizing algebra is non-amenable and therefore by Theorem 6.1 we can assume that  $C \preceq_M M_1$ . Therefore one can find nonzero projections  $p \in C$ ,  $q \in M_1$ , a partial isometry  $v \in M$ , and a  $\star$ -homomorphism  $\phi : Cp \rightarrow qM_1q$  such that for all  $x \in Cp$  we have

$$(22) \quad \phi(x)v = vx.$$

Since  $C$  is a maximal abelian subalgebra of  $M$  then by Lemma 1.5 in [24] we can assume that  $\phi(Cp) \subset qM_1q$  is also a maximal abelian subalgebra. Fixing  $u \in \mathcal{N}_{Mp}(Cp)$  we can easily see that for all  $x \in Cp$  we have

$$(23) \quad vuv^*\phi(x) = vuv^*vxv^* = vuv^*uxu^*v^* = \phi(uxu^*)vuv^*.$$

Notice that  $vuv^*vu^*v^* = \phi(uv^*vu^*)v^*$  is a projection and hence  $vuv^*$  is a partial isometry. Also, applying the conditional expectation  $E_{qM_1q}$  to equation (23), we obtain that for all  $x \in Cp$  we have

$$E_{qM_1q}(vuv^*)\phi(x) = \phi(uxu^*)E_{qM_1q}(vuv^*).$$

Taking the polar decomposition  $E_{qM_1q}(vuv^*) = w_u|E_{qM_1q}(vuv^*)|$ , the previous equation entails that  $|E_{qM_1q}(vuv^*)| \in \phi(Cp)' \cap qM_1q = \phi(Cp)$  and for all  $x \in Cp$  we have

$$w_u\phi(x) = \phi(uxu^*)w_u.$$

This implies in particular that  $w_uw_u^*, w_u^*w_u \in \phi(Cp)' \cap qM_1q = \phi(Cp)$  and therefore  $w_u \in \mathcal{GN}_{qM_1q}(\phi(Cp))$ , the normalizing groupoid of  $\phi(Cp)$  in  $qM_1q$ . Altogether, we have shown that

$$E_{qM_1q}(vuv^*) \subseteq \mathcal{GN}_{qM_1q}(\phi(Cp))'' = \mathcal{N}_{qM_1q}(\phi(Cp))''.$$

Since the above containment holds for every  $u \in \mathcal{N}_{pMp}(Cp)''$  and  $\mathcal{N}_{pMp}(Cp)'' = pMp$  we have that

$$E_{qM_1q}(vMv^*) \subseteq \mathcal{N}_{qM_1q}(\phi(Cp))'',$$

and hence  $vv^*M_1vv^* \subseteq \mathcal{N}_{qM_1q}(\phi(Cp))''$ . This shows in particular that  $\mathcal{N}_{qM_1q}(\phi(Cp))''$  is non-amenable; therefore, by Theorem B in [9] we have that  $\phi(Cp) \preceq_{M_1} A$ . By Remark 3.8 in [63] this further implies that  $C \preceq_M A$ . Finally, by Theorem 1.4, one can find a unitary  $x \in \mathcal{U}(M)$  such that  $x\phi(B)x^* = xCx^* = A$ .

In particular, our claim shows that the actions  $\Gamma_1 \times \Gamma_2 \curvearrowright X$  and  $\Lambda \curvearrowright Y$  are orbit equivalent. Note that, since  $\Gamma_1$  and  $\Gamma_2$  have property (T) then so is the product  $\Gamma_1 \times \Gamma_2$ , so it follows from Ioana's Cocycle Superrigidity Theorem [22] that the actions  $\Gamma_1 \times \Gamma_2 \curvearrowright X$  and  $\Lambda \curvearrowright Y$  are virtually conjugate.  $\square$

**Corollary 6.5.** *Let  $\Gamma_1$  and  $\Gamma_2$  be nonamenable hyperbolic groups. If  $\Lambda$  is measure equivalent to  $\Gamma_1 \times \Gamma_2$  then  $\Lambda$  does not have infinite amenable normal subgroups.*

*Proof.* We will assume that  $\Lambda$  is a group which admits an amenable normal subgroup  $\Sigma$  such that  $\Lambda \cong_{ME} \Gamma_1 \times \Gamma_2$  and that show this leads to a contradiction. By a Lemma 3.2 and Theorem 3.3 in [14], this implies that there exist stably-orbit equivalent free, ergodic actions  $\Gamma_1 \times \Gamma_2 \curvearrowright X$  and  $\Lambda \curvearrowright Y$ . For simplicity, we will assume that the two actions are orbit equivalent, i.e. there exists a  $\star$ -isomorphism  $\psi : L^\infty(Y) \rtimes \Lambda \rightarrow L^\infty(X) \rtimes (\Gamma_1 \times \Gamma_2)$ . We will also denote by  $A = L^\infty(X)$ ,  $B = L^\infty(Y)$ ,  $P = \psi(L^\infty(Y) \rtimes \Sigma)$ ,  $M = A \rtimes (\Gamma_1 \times \Gamma_2)$ ,  $M_1 = A \rtimes \Gamma_1$ ,  $M_2 = A \rtimes \Gamma_2$ , and notice that  $\psi(B) = A$ .

Since the Cowling-Haagerup constant is an  $ME$ -invariant it follows that  $\psi(L\Sigma)$  is a weakly compact embedding in  $M$ . Since  $\Sigma$  is normal in  $\Lambda$ , then applying Theorem 6.1, we can assume that  $\psi(L\Sigma) \preceq_M M_1$  and since  $\psi(B) = A$  we conclude that  $P \preceq_M M_1$ . Therefore, one can find nonzero projections  $p \in P$ ,  $q \in M_1$ , a partial isometry  $v \in M$ , and a  $\star$ -homomorphism  $\phi : pPp \rightarrow qM_1q$  such that for all  $x \in pPp$  we have

$$(24) \quad \phi(x)v = vx.$$

Since  $P$  is a irreducible subfactor of  $M$ , by Proposition 1.2 we can assume that  $\phi(pPp) \subset qM_1q$  is also a irreducible subfactor. Fixing  $u \in \mathcal{N}_{pMp}(pPp)$  we can easily see that for all  $x \in pPp$  we have

$$(25) \quad vuv^*\phi(x) = vuv^*v^* = vuv^*u^*v^* = \phi(uxu^*)vuv^*.$$

Notice that  $vuv^*v^*v^* = \phi(uv^*vu^*)v^*$  is a projection and hence  $vuv^*$  is a partial isometry. Also, applying the conditional expectation  $E_{qM_1q}$  to equation (25), we obtain that for all  $x \in pPp$  we have

$$E_{qM_1q}(vuv^*)\phi(x) = \phi(uxu^*)E_{qM_1q}(vuv^*).$$

Taking the polar decomposition  $E_{qM_1q}(vuv^*) = w_u|E_{qM_1q}(vuv^*)|$ , the previous equation entails that  $|E_{qM_1q}(vuv^*)| \in \phi(pPp)' \cap qM_1q = \mathbb{C}q$  and for all  $x \in pPp$  we have

$$w_u\phi(x) = \phi(uxu^*)w_u.$$

This implies in particular that  $w_uw_u^*, w_u^*w_u \in \phi(pPp)' \cap qM_1q = \mathbb{C}q$  and therefore  $w_u$  is a scalar multiple of a normalizing unitary in  $\mathcal{N}_{qM_1q}(\phi(pPp))$ . Altogether, we have shown that

$$E_{qM_1q}(vuv^*) \subseteq \mathcal{N}_{qM_1q}(\phi(pPp))''.$$

Since the above containment holds for every  $u \in \mathcal{N}_{pMp}(pPp)$  and  $\mathcal{N}_{pMp}(pPp)'' = pMp$  we have that

$$E_{qM_1q}(vMv^*) \subseteq \mathcal{N}_{qM_1q}(\phi(pPp))'',$$

and hence  $vv^*M_1vv^* \subseteq \mathcal{N}_{qM_1q}(\phi(pPp))''$ . This shows in particular that  $\mathcal{N}_{qM_1q}(\phi(pPp))''$  is non-amenable; therefore, by Theorem B in [9] we have that  $\phi(pPp) \preceq_{M_1} A$ . By Remark 3.8 in [63] this would imply that  $P \preceq_M A$ , which is an obvious contradiction.  $\square$

## 7. FURTHER RESULTS AND FINAL REMARKS

The main purpose of this section is to point out that many of the structural results obtained in [9] can be pushed forward in the context of groups which admit arrays that are proper with respect to certain families of subgroups. For instance, we have the following result is a generalization of Theorem 3.2 and Theorem 4.1 from [9]. Our proof largely overlaps with the proof of Theorem 4.1. [9], but there is a key step which will allow us to upgrade the result from quasi-cocycles as presented

in [9] to arrays. We include a proof below which addresses only this step. The technical argument used is essentially the same as in the Lemma 6.2 above in the case of a single factor. We reproduce it here for the reader's convenience .

**Theorem 7.1.** *Let  $\Gamma$  be an exact group together with a family of subgroups  $\mathcal{G}$ , and let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be a weakly- $\ell^2$  representation. Also, let  $\Gamma \curvearrowright X$  be a free, ergodic action and denote by  $M = L^\infty(X) \rtimes \Gamma$  the corresponding crossed-product von Neumann algebra.*

1. *If  $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$  and  $P \subseteq M$  is diffuse subalgebra, then either  $A' \cap M$  is amenable or there exists a group  $\Sigma \in \mathcal{F}$  such that  $P \preceq_M L^\infty(X) \rtimes \Sigma$ .*
2. *If  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$  and  $P \subseteq M$  is a weakly compact embedding with  $P$  diffuse, then either the normalizing algebra  $\mathcal{N}_M(P)''$  is amenable or there exists  $\Sigma \in \mathcal{G}$  such that  $P \preceq_M L^\infty(X) \rtimes \Sigma$ .*
3. *Assume that  $\mathcal{G}$  is a family of quasi-normal subgroups of  $\Gamma$ . If  $\mathcal{RA}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$  and  $P \subseteq M$  is a weakly compact embedding with  $P$  diffuse, then either the normalizing algebra  $\mathcal{N}_M(P)''$  is amenable or there exists  $\Sigma \in \mathcal{G}$  such that  $P \preceq_M L^\infty(X) \rtimes \Sigma$ .*

*Proof.* As stated, the first part is Theorem 3.2 in [9] while the second part follows exactly as in the proof of Theorem 4.1 in [9]. Indeed the only ingredient needed for this is to adapt Proposition 2.6 in [9] to the case of quasi-cocycles that are proper with respect to a family of subgroups. One can see however that this is a straight forward exercise and we leave it to the reader. So we only prove the third part.

For simplicity we will denote by  $N = \mathcal{N}_M(P)''$  and then we fix  $p \in \mathcal{Z}(N' \cap M)$  a projection. Also to not complicate the notations we assume that  $\mathcal{G}$  consists of a single subgroup of  $\Gamma$ , i.e.,  $\mathcal{G} = \{\{\Sigma\}\}$ . Then the general strategy of the proof is to show that the assumption  $P \not\preceq_M L^\infty(X) \rtimes \Sigma$  implies that  $Np$  is amenable. By assumption  $P \subset M$  is weakly compact, so there exists a net of positive unit vectors  $(\eta_n)_{n \in \mathbf{N}}$  in  $L^2(M) \otimes L^2(M)$  such that

- (1)  $\|\eta_n - (v \otimes \bar{v})\eta_n\| \rightarrow 0$ , for all  $v \in \mathcal{U}(P)$ ;
- (2)  $\|[u \otimes \bar{u}, \eta_n]\| \rightarrow 0$ , for all  $u \in \mathcal{N}_M(P)$ ; and
- (3)  $\langle (x \otimes 1)\eta_n, \eta_n \rangle = \tau(x) = \langle (1 \otimes \bar{x})\eta_n, \eta_n \rangle$ , for all  $x \in M$ .

From here on the proof follows the same argument as in Lemma 6.2 in the previous section. So let  $\mathcal{H} = L_0^2(Y^\pi) \otimes L^2(X) \otimes \ell^2(\Gamma)$  which as we remarked before is weakly contained as an  $M$ -bimodule in the coarse bimodule. Fixing  $t > 0$  we consider the unitary  $V_t$  associated with an array  $q$  as defined in the previous sections. Next denote by  $\tilde{\eta}_{n,t} = (V_t \otimes 1)(p \otimes 1)\eta_n$ ,  $\zeta_{n,t} = (e \otimes 1)\tilde{\eta}_{n,t} = (e \cdot V_t \otimes 1)(p \otimes 1)\eta_n$ , and  $\xi_{n,t} = \tilde{\eta}_{n,t} - \zeta_{n,t} = (e^\perp \otimes 1)\tilde{\eta}_{n,t} \in \mathcal{H} \otimes L^2(M)$ .

Using these notations we show next the following inequality:

**Lemma 7.2.**

$$\lim_n \|\xi_{n,t}\| \geq \frac{1}{16} \|p\|_2.$$

*Proof.* We argue by contradiction, so passing to a subsequence we assume that

$$(26) \quad \|\xi_{n,t}\| < \frac{1}{16} \|p\|_2 \text{ for all } n.$$

Denoting by  $\zeta_n = (p \otimes 1)\eta_n$  we have  $\|\tilde{\eta}_{n,t}\| = \|\zeta_n\| = \|p\|_2$  and using the identity  $\|(e \otimes 1)(\tilde{\eta}_{n,t})\|^2 + \|(e^\perp \otimes 1)(\tilde{\eta}_{n,t})\|^2 = \|\tilde{\eta}_{n,t}\|^2 = \|p\|_2^2$  in combination with (26) we

have

$$(27) \quad \|(e \otimes 1)(\tilde{\eta}_{n,t})\| > \frac{15}{16}\|p\|_2, \text{ for all } n.$$

Throughout the proof, for any subset  $\mathcal{F} \subset \Gamma$ , we denote by  $P_{\mathcal{F}}$  the orthogonal projection from  $L^2(M) \bar{\otimes} L^2(\bar{M})$  onto the closed linear span of the set  $\{((L^\infty(X) \rtimes \Sigma)u_h) \bar{\otimes} \bar{M} \mid h \in \mathcal{F}\}$ .

The main strategy is to prove that relation (27) together with the assumption  $Pp \not\leq_M L^\infty(X) \rtimes \Sigma$  will enable us to construct an infinite sequence  $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \dots$  of finite subsets of  $\Gamma$  such that  $\Sigma\mathcal{F}_i$  are disjoint and which satisfies the following property: for every  $k \in \mathbb{N}$  we can find  $n_k \in \mathbb{N}$  such that for all  $i \leq k$  and  $n \geq n_k$  we have

$$(28) \quad \|P_{\mathcal{F}_i}(\zeta_n)\| \geq \frac{1}{1000}\|p\|_2.$$

First we briefly explain how this claim leads to a contradiction, thus finishing the proof of the lemma. Since the sets  $\mathcal{F}_i \Sigma\mathcal{F}_i$  are disjoint, relation (28) implies  $\|p\|_2^2 = \|\zeta_n\|^2 \geq \sum_{i=1}^k \|P_{\mathcal{F}_i}(\zeta_n)\|_2^2 \geq k \left( \frac{1}{1000}\|p\|_2 \right)^2$ , for all  $k \in \mathbb{N}$  and  $n \geq n_k$ . This is obviously impossible when letting  $k$  be sufficiently large.

So we are left to prove (28). To show this we will proceed by induction on  $k$ .

First we prove case  $k = 1$ . Since  $\zeta_n \in L^2(M) \bar{\otimes} L^2(\bar{M})$ , we write  $\zeta_n = \sum_{g \in \Gamma} \zeta_g^n \delta_g$ , where  $\zeta_g^n \in L^2(X) \bar{\otimes} L^2(\bar{M})$ . Then, using the definition of  $V_t$ , a straight forward computation shows that

$$\|e(\tilde{\eta}_{n,t})\|^2 = \sum_{g \in \Gamma} \exp(-2t^2\|q(g)\|^2) \|\zeta_g^n\|^2, \text{ for all } n.$$

When combined with (27) this formula implies that, for all  $n$  we have

$$(29) \quad \sum_{g \in \Gamma} \exp(-2t^2\|q(g)\|^2) \|\zeta_g^n\|^2 > \frac{211}{256}\|p\|_2^2.$$

Since the map  $g \rightarrow \|q(g)\|^2$  is a proper relatively to  $\{\Sigma\}$ , then the set  $\{g \in \Gamma \mid \exp(-t^2\|q(g)\|^2) \leq \frac{1}{2}\}$  is contained in  $\Sigma\mathcal{F}$  for some finite set  $\mathcal{F} \subset \Gamma$  and, using the inequality (29), we further deduce that

$$\frac{211}{256}\|p\|_2^2 < \frac{1}{4} \sum_{g \in \Gamma \setminus \Sigma\mathcal{F}} \|\zeta_g^n\|^2 + \sum_{g \in \Sigma\mathcal{F}} \|\zeta_g^n\|^2, \text{ for all } n.$$

By basic algebraic manipulations, the above inequality gives that  $\sum_{g \in \Sigma\mathcal{F}} \|\zeta_g^n\|^2 >$

$\frac{49}{64}\|p\|_2^2$  for all  $n$  which implies

$$(30) \quad \|P_{\mathcal{F}}(\zeta_n)\| > \frac{7}{8}\|p\|_2.$$

So case  $k = 1$  follows by letting  $\mathcal{F}_1 = \mathcal{F}$  and  $n_1 = 1$ .

Next we show the induction step, i.e., assuming that we have constructed the sets  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k \subset \Gamma$  and  $n_k \in \mathbb{N}$ , we indicate how to construct  $\mathcal{F}_{k+1} \subset \Gamma$  and  $n_{k+1} \in \mathbb{N}$  satisfying (28).

Consider the set  $\mathcal{G} = \cup_{i=1}^k (\mathcal{F}_i \mathcal{F}^{-1}) \subset \Gamma$ . Since  $\mathcal{G}$  is finite and  $Pp \not\prec_M L^\infty(X)$ , by Popa's intertwining techniques, there exist a unitary  $v \in \mathcal{U}(P)$ , a finite set  $\mathcal{K} \subset \Gamma$ , and an element  $v'$  in the linear span of  $\{(L^\infty(X) \rtimes \Sigma)u_h \mid h \in \mathcal{K}\}$  such that

$$(31) \quad \Sigma\mathcal{K} \cap \Sigma\mathcal{G} = \emptyset;$$

$$(32) \quad \|v' - vp\|_2 \leq \frac{1}{1000|\mathcal{F}|} \|p\|_2.$$

Next, we show that for  $n \in \mathbb{N}$  and  $z \in M$  we have

$$(33) \quad \|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq |\mathcal{F}| \|z\|_2.$$

Fix  $n$  and denote by  $P$  the orthogonal projection onto  $L^2(X) \bar{\otimes} L^2(\bar{M})$ , i.e.  $P = P_\emptyset$ . We have  $P_{\mathcal{F}}(\zeta_n) = \sum_{h \in \mathcal{F}} P(\zeta_n(u_h^* \otimes 1))(u_h \otimes 1)$  and by the Cauchy-Schwarz inequality we deduce

$$(34) \quad \|(z \otimes 1)P_{\mathcal{F}}(\zeta_n)\|^2 \leq |\mathcal{F}| \sum_{h \in \mathcal{F}} \|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2$$

Now we let  $E_{L^\infty(X)}$  to be the conditional expectation from  $M$  onto  $L^\infty(X)$  and we denote by  $a = E_{L^\infty(X)}(z^* z)^{\frac{1}{2}}$ . Using the formulas  $\langle (x \otimes 1)P(\zeta), P(\zeta) \rangle = \langle (E_{L^\infty(X)}(x) \otimes 1)P(\zeta), P(\zeta) \rangle$  and  $\|(x \otimes 1)\eta_n\| = \|x\|_2$ , for all  $\zeta \in L^2(M) \bar{\otimes} L^2(\bar{M})$  and  $x \in M$ , we obtain the following:

$$\begin{aligned} & \|(z \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|^2 \\ &= \langle (z^* z \otimes 1)P(\zeta_n(u_h^* \otimes 1)), P(\zeta_n(u_h^* \otimes 1)) \rangle \\ &= \langle (E_{L^\infty(X)}(z^* z) \otimes 1)P(\zeta_n(u_h^* \otimes 1)), P(\zeta_n(u_h^* \otimes 1)) \rangle \\ &= \|(a \otimes 1)P(\zeta_n(u_h^* \otimes 1))\|_2^2 = \|P((a \otimes 1)\zeta_n(u_h^* \otimes 1))\|^2 \\ &\leq \|(a \otimes 1)\zeta_n\|^2 = \|ap\|_2^2 \leq \|a\|_2^2 = \|z\|_2^2. \end{aligned}$$

It is clear that the last inequalities combined with (34) give (33).

To this end, applying the triangle inequality, for all  $v \in \mathcal{U}(P)$  and all  $n \in \mathbb{N}$ , we have

$$(35) \quad \begin{aligned} & \|\zeta_n - (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \\ & \|\zeta_n - (vp \otimes \bar{v})\zeta_n\| + \|\zeta_n - P_{\mathcal{F}}(\zeta_n)\| + \|((v' - vp) \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \end{aligned}$$

Since  $p$  and  $v$  commute, we have  $\zeta_n - (vp \otimes \bar{v})\zeta_n = (p \otimes 1)(\eta_n - (v \otimes \bar{v})\eta_n)$ . Thus, since  $\lim_{n \rightarrow \infty} \|\eta_n - (v \otimes \bar{v})\eta_n\|_2 = 0$ , we can find  $n_{k+1} \geq n_k$  such that for all  $n \geq n_{k+1}$  we have

$$(36) \quad \|\zeta_n - (vp \otimes \bar{v})\zeta_n\| \leq \frac{1}{1000} \|p\|_2.$$

Using (33) for  $z = vp - v'$  in combination with (32) for all  $n$  we have

$$(37) \quad \|((v' - vp) \otimes 1)P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{1000} \|p\|_2.$$

Altogether, (35), (36), (37), and (30) show that that for all  $n \geq n_{k+1}$  we have

$$(38) \quad \|\zeta_n - (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)\| \leq \frac{1}{500} \|p\|_2 + \|(\zeta_n - P_{\mathcal{F}}(\zeta_n))\| < \frac{8 + 500\sqrt{15}}{4000} \|p\|_2.$$



Finally we let  $\mathcal{F}_{k+1} = \mathcal{K}\mathcal{F}$  and by (31) we see that  $\Sigma\mathcal{F}_{k+1}$  is disjoint from  $\Sigma\mathcal{F}, \Sigma\mathcal{F}_2, \dots, \Sigma\mathcal{F}_k$ . Moreover, we have that  $(v' \otimes v)P_{\mathcal{F}}(\zeta_n)$  belongs to the closed linear span of  $\{(L^\infty(X)u_h) \otimes \bar{M} \mid h \in \mathcal{F}_{k+1}\}$ . Thus,  $P_{\mathcal{F}_{k+1}}((v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)) = (v' \otimes \bar{v})P_{\mathcal{F}}(\zeta_n)$  and (38) together with the triangle inequality give that

$$\|\zeta_n - P_{\mathcal{F}_{k+1}}(\zeta_n)\| < \frac{8 + 500\sqrt{15}}{2000} \|p\|_2, \text{ for all } n \geq n_{k+1}.$$

Hence, for all  $n \geq n_{k+1}$ , we have

$$\|P_{\mathcal{F}_{k+1}}(\zeta_n)\| > \left(1 - \left(\frac{8 + 500\sqrt{15}}{2000}\right)^2\right)^{\frac{1}{2}} \|p\|_2 > \frac{1}{1000} \|p\|_2,$$

which ends the proof of (28).  $\square$

Next we briefly explain how to use the previous lemma in order to get the proof of the theorem. First notice that, as an  $M$ -bimodule,  $\mathcal{H}$  is weakly contained in the coarse bimodule. Then following the same argument as in Theorem B of [42] we define a state  $\psi_t$  on  $\mathcal{N} = \mathfrak{B}(\mathcal{H}) \cap \rho(M^{op})'$ . Explicitly, if we denote by  $\xi_{n,t} = e(\tilde{\eta}_{n,t})$  we let  $\psi_t(x) = \lim_n \frac{1}{\|\xi_{n,t}\|_2} \langle (x \otimes 1)\xi_{n,t}, \xi_{n,t} \rangle$  for every  $x \in \mathcal{N}$ .

To get the proof, from here on, one can proceed exactly as explained in Theorem 4.1 in [9] or Theorem 6.1 from the previous section. Namely we use the same Lemmas 4.3 and 4.4 from [9] and the final part in the proof of Theorem B in [42] to conclude that  $Np$  is amenable. We leave the details to the reader.  $\square$

The first part of the conclusion is a restatement (via Proposition 2.1) of a well-known theorem due to Ozawa (Theorem in [5]), our contribution being the second and the third part. We should mention that in the light of Proposition 1.10 in [9] it is not very difficult to see that there is a way to deduce Theorem 6.1 from the third part of Theorem 7.1.

Finally, we note that very recently Popa and Vaes extended the third part to arbitrary families of subgroups, or in the case that  $\Gamma$  is weakly amenable, arbitrary free ergodic actions [54].

In the remaining part of the section we explain how the second and third part can be successfully exploited to produce new examples of von Neumann algebras with either unique Cartan subalgebra or no Cartan subalgebras. With this purpose in mind, we introduce the following definition.

**Definition 7.3.** A subgroup  $\Sigma < \Gamma$  is called *weakly malnormal* if there exist finitely many elements  $\gamma_1, \gamma_2, \dots, \gamma_n \in \Gamma$  such that

$$\left| \bigcap_{i=1}^n \gamma_i \Sigma \gamma_i^{-1} \right| < \infty.$$

Therefore, when the second and the third part in the intertwining theorem above is combined with Corollary 7 from [19] we immediately obtain the following uniqueness (absence) of Cartan subalgebra statement.

**Corollary 7.4.** Let  $\Gamma$  be a weakly amenable group and let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be an weakly- $\ell^2$  representation such that one of the following cases holds:

- (1)  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$  for a family of weakly malnormal subgroups  $\mathcal{G}$  of  $\Gamma$ , or

(2)  $\mathcal{RA}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ .

Also let  $\Gamma \curvearrowright X$  be a weakly compact, free action. If  $\Gamma$  is as in the first case (1) above we assume in addition that the restrictions  $\Sigma \curvearrowright X$  are ergodic for all  $\Sigma \in \mathcal{G}$ .

Then  $L^\infty(X) \rtimes \Gamma$  has unique Cartan subalgebra. If in addition  $\Gamma$  is i.c.c. and  $\mathcal{G}$  is a family of malnormal groups then  $L\Gamma$  has no Cartan subalgebra.

Employing the same strategy as in the proof of Corollary B.2 from [9] and using the fact that the class of weakly amenable groups is closed under ME-subgroups, we obtain new structural results for measure equivalence of groups.

**Corollary 7.5.** *Let  $\Gamma$  be a weakly amenable group and let  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  be an weakly- $\ell^2$  representation such that one of the following holds: either  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$  for a family of amenable, malnormal subgroups  $\mathcal{G}$ , or  $\mathcal{RQ}(\Gamma, \{e\}, \mathcal{H}_\pi) \neq \emptyset$ . If  $\Lambda$  is any ME-subgroup of  $\Gamma$  then  $L\Lambda$  is strongly solid i.e., given any diffuse amenable subalgebra  $A \subseteq L\Lambda$  its normalizing algebra  $\mathcal{N}_{L\Lambda}(A)''$  is still amenable. In particular, every amenable subgroup of  $\Lambda$  has amenable normalizer.*

**Example 7.6.** The following are examples of groups that satisfy the conditions required in the above corollary: any weakly amenable group that is in the class  $\mathcal{S}$  of Ozawa [36]—in particular any weakly amenable group  $\Gamma$  that is hyperbolic relative to a family of amenable subgroups (e.g. Sela’s limit groups which are weakly amenable and hyperbolic with respect to their noncyclic maximal abelian subgroups [11]); any weakly amenable HNN extension  $\Gamma \star_\alpha$  of a group  $\Gamma$ , where  $\alpha : \Sigma_1 \rightarrow \Sigma_2$  is a monomorphism with  $\Sigma_i \in \mathcal{G}$ ; any infinite free product  $\star_{n \in \mathbb{N}} \Gamma_n$  where  $\Gamma_n$  is hyperbolic relative to a finite family  $\mathcal{G}_n$  of malnormal groups,  $\Lambda_{cb}(\Gamma_n) = 1$  and  $\mathcal{G}_n = \{e\}$  for all but finitely many  $n$ ’s — in this case we choose  $\mathcal{G} = \cup_n \mathcal{G}_n$ .

Next we discuss another application of the main intertwining theorem: let  $\Gamma$  be a hyperbolic group and let  $\Lambda$  be an abelian inductive limit of finite groups. Assume that  $\rho : \Lambda \rightarrow \text{Aut}(\Gamma)$  is an action by automorphism with finite orbits such that for every  $\lambda \in \Lambda$  and every finite index subgroup  $\Gamma_0 < \Gamma$  the restriction  $\rho_\lambda|_{\Gamma_0}$  is not the conjugacy by an element in  $\Gamma$ . Then applying part (2) in Theorem 7.1 we obtain that every type II, amenable subalgebra  $A \subset L(\Gamma \rtimes \Lambda)$  has amenable normalizing algebra  $\mathcal{N}_{L(\Gamma \rtimes \Lambda)}(A)''$ . Also if we assume that  $A \subset L(\Gamma \rtimes \Lambda)$  is a semiregular MASA containing  $L\Lambda$  with non-amenable normalizing algebra then any other semiregular MASA in  $L(\Gamma \rtimes \Lambda)$  is either unitary conjugated to  $A$  or it has amenable normalizing algebra. Finally, we notice that if we assume  $\Lambda$  is i.c.c. and  $L(\Lambda)' \cap L(\Gamma \rtimes \Lambda)$  has no amenable summand then every MASA in  $L(\Gamma \rtimes \Lambda)$  have amenable normalizing algebra.

In the remaining part of the section, we discuss some insight provided by Theorem 7.1 regarding the structure of equivalence relations induced by certain p.m.p. actions. Suppose  $\Gamma \curvearrowright X$  is any free ergodic action with  $\Gamma$  satisfying the hypothesis from part (1) of Theorem 7.1 and let  $\mathcal{G}$  be a family of weakly malnormal subgroups of  $\Gamma$ . If  $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$  denotes the equivalence relation on  $X \times X$  induced by  $\Gamma \curvearrowright X$  then  $\mathcal{R} \not\cong \mathcal{R}_{\Lambda_1 \times \Lambda_2 \curvearrowright Y}$ , for any free action  $\Lambda_1 \times \Lambda_2 \curvearrowright Y$  with  $\Lambda_i$  infinite.

To see this, we assume by contradiction that there exists an  $\star$ -isomorphism between the von Neumann algebras  $\phi : L^\infty(Y) \rtimes (\Lambda_1 \times \Lambda_2) \rightarrow L^\infty(X) \rtimes \Gamma = M$  which preserves the Cartan subalgebras. Since  $\Gamma$  is non-amenable, we may assume without loosing any generality that  $\Lambda_1$  is also non-amenable. Then applying first part of Theorem 7.1 we have that  $\phi(L\Lambda_2) \preceq_M L^\infty(X) \rtimes \Sigma$ , for some  $\Sigma \in \mathcal{G}$ . Since  $\phi$

preserves the Cartan subalgebras, we also have that  $\phi(L^\infty(X) \rtimes \Lambda_2) \preceq_M L^\infty(X) \rtimes \Sigma$ . Furthermore, since  $\phi(L^\infty(X) \rtimes \Lambda_2)$  is regular in  $M$  and  $\Sigma$  is weakly malnormal in  $\Gamma$ , then Corollary 7 in [19] implies that  $\phi(L^\infty(X) \rtimes \Lambda_2) \preceq_M L^\infty(X)$ , which is obviously a contradiction because  $\Lambda_2$  is infinite.

This expands upon some indecomposability results for equivalence relations from [1]. We mention that this result also follows from Ozawa's earlier results, [5]. It is very likely that this indecomposability property still holds without the exactness assumption on  $\Gamma$ . In this case however the proof should be more ergodic theoretic in nature, as the present von Neumann algebras techniques rely heavily on exactness.

If in addition we assume that the action  $\Gamma \curvearrowright X$  is weakly compact and  $\Gamma$  is hyperbolic group with  $\mathcal{G} = \{e\}$  then by the second part in Theorem 7.1 we have the following: for any infinite, amenable sub-equivalence relation  $\mathcal{S} \subset \mathcal{R}$  its normalizing equivalence relation  $\mathcal{N}_{\mathcal{R}}(\mathcal{S})$  in  $\mathcal{R}$  is still amenable. Notice that this partially recovers Corollary 6.2 from [1]. However, there is also a more general version of this result.

**Corollary 7.7.** *Let  $\Gamma$  be a weakly amenable group together with a family of malnormal subgroups  $\mathcal{G}$  and a weakly- $\ell^2$  representation  $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H}_\pi)$  such that  $\mathcal{RQ}(\Gamma, \mathcal{G}, \mathcal{H}_\pi) \neq \emptyset$ . Let  $\Gamma \curvearrowright X$  be a weakly compact, free, ergodic, p.m.p. action and denote by  $\mathcal{R}$  the induced equivalence relation. Then given any free infinite, amenable sub-equivalence relation  $\mathcal{S} \subset \mathcal{R}$  its normalizing equivalence relation  $\mathcal{N}_{\mathcal{R}}(\mathcal{S})$  in  $\mathcal{R}$  is still amenable.*

The proof follows from Theorem 7.1 and Proposition 1.3 by applying the same argument as in the proof of Corollary 0.5.

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